Relativistic Kinematics

6.1 TIME DILATION, LENGTH CONTRACTION AND SIMULTANEITY

In the next section we shall find the new equations which will replace the Galilean transformation equations (5.1) and (5.2), but before that let us derive perhaps the two most remarkable results in Einstein's theory: the fact that time passes at different rates in different inertial frames and that it doesn't make sense to speak of the length of a metre rule without also stating the frame in which it is at rest.

Historically people have regarded distance and time as fundamental units. For example, as defined by a standard length of material and an accurate periodic device. Speed is then a derived quantity determined by the ratio of distance travelled and time taken. Nowadays, the scientific community has stopped thinking of the metre as fundamental. Instead the metre is defined to be the distance travelled in a vacuum by light in a time of exactly 1/2,9979,2458 seconds. This might look like a rather arbitrary definition but that particular sequence of numbers in the denominator means that the metre so defined corresponds to the length of the old standard metre, which was a metal bar kept locked in a vault in Paris. The advantage of defining the metre in terms of the speed of light and the unit of time means that we no longer have to worry about the fact that the metal bar is forever changing as it expands and contracts. By defining the metre this way we have chosen a value for the speed of light in a vacuum, i.e. $c = 2.99792458 \times 10^8$ m/s. There is nothing particularly special about using the speed of light here, strictly speaking one could define the metre to be the distance travelled by an average snail in 15 minutes. Then the snail speed would be fundamental. However, given the variability in snail speeds, this would not consitute a very reliable measure. Light speed is much more preferable and it has the particular advantage that it is the only speed which everyone agrees upon (by Einstein's 2nd postulate); all other speeds require the specification of an associated frame of reference.

Dynamics and Relativity Jeffrey R. Forshaw and A. Gavin Smith

Although this definition of distance suits most people, it isn't really the best definition for physicists who work with particles travelling close to light speed. As a result, the metre is sometimes rejected in favour of a distance measure such that 1 unit of distance is equal to the distance travelled by light in 1 second. In these units, which particle physicists prefer, $c = 1$.

6.1.1 Time Dilation and the Doppler Effect

Conversely, one could define time by specifying a speed and a distance. For example, we could make a clock by bouncing light between two mirrors spaced by a known distance, as illustrated in Figure 6.1. We can think of one 'tick' of this clock as corresponding to the time it takes the light to travel between the two mirrors and back. The time interval between any two events can then be determined by counting the number of 'ticks' of the light-clock which have elapsed between the two events. Of course there is nothing special about light here, for example we could define time by bouncing a ball between two walls.

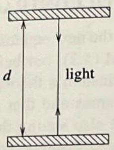

Figure 6.1 A light-clock viewed in its rest frame.

This is a good place to discuss exactly how time measurements are to be made. Consider an observer in some frame of reference S who is interested in making some time measurements. Since Einstein's theory is going to require that we drop the notion of absolute time, we need to be more careful than usual in specifying how the time of an event is determined. Ideally, the observer would like to have a set of identical clocks all at rest in S with one clock at each point in space. For convenience, the observer might choose that the clocks are all synchronised with each other. The time of an event is then determined by the time registered on a clock close to the event. Ideally the clock would be at the same place as the event otherwise we should worry about just how the information travels from the event to the clock. The observer can then determine the time of an event by travelling to the clock co-incident with the event and reading the time at which the event occured (we are imagining that the clock was stopped by the event and the time recorded). Clearly this is not a very practicable way of measuring the time of an event but that is not the point. We have succeeded in explaining in principle what we mean by the time of an event. Most importantly, the time of the event clearly has nothing to do with where the observer was when the event happened nor whether the observer actually saw the event with their eyes. We may have laboured this point to excess

but that is because there is room for much confusion if these ideas are not properly appreciated.

Let us return to the light-clock of Figure 6.1. In its rest frame, the time it takes for light to do the roundtrip between the mirrors (one 'tick') is clearly

$$\Delta t_0 = \frac{2d}{c}. \tag{6.1}$$

Now let us imagine what happens if the clock is moving relative to the observer. To be specific let us put the clock in S' and an observer in S where the two frames are as usual defined by Figure 5.1. If the observer was in S' then the time for one tick of the clock would be just Δt_0. Our task is to determine the corresponding time when the observer is in S. According to this observer, the light follows the path shown in Figure 6.2. We call Δt the time it takes for the light to complete one roundtrip as measured in S. Accordingly the clock moves a distance $x_2 - x_1 = v\Delta t$ over the course of the roundtrip. Using Pythagoras' Theorem, it follows that the light travels a total distance $2(d^2 + v^2\Delta t^2/4)^{1/2}$. All of this is as it would be in Galilean relativity. Now here comes the new idea. The light is still travelling at speed c in S (in classical theory the speed would be $(c^2 + v^2)^{1/2}$ by the simple addition of velocities). As a result, the time for the roundtrip in S satisfies

$$\Delta t = \frac{2}{c}\left(d^2 + \frac{v^2\Delta t^2}{4}\right)^{1/2}. \tag{6.2}$$

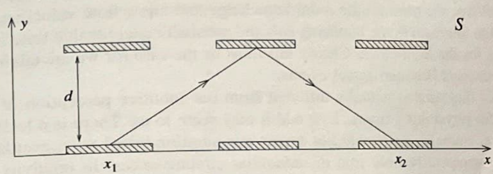

Figure 6.2 The path taken by the light in a moving light-clock.

Squaring both sides and re-arranging allows us to solve for Δt:

$$\Delta t = \frac{2d}{c} \times \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{6.3}$$

The time measured in S is longer than the time measured in S' and we are forced to conclude that in Einstein's theory *moving clocks run slow*. This effect is also known as 'time dilation', and it is negligibly small if $v/c \ll 1$ but when $v \sim c$ the effect is dramatic.

The factor $1/\sqrt{1-v^2/c^2}$ appears so often in Special Relativity that it is given its own symbol, i.e.

$$\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} \tag{6.4}$$

and

$$\Delta t = \gamma \Delta t_0. \tag{6.5}$$

For $v/c \le 1$ it follows that $\gamma > 1$ and for $v/c > 1$ the theory doesn't appear to make much sense (unless we are prepared to entertain the idea of imaginary time).

To conclude this section, let us quickly check that $\Delta t = \Delta t_0$ in classical theory. Replacing c in Eq. (6.2) by $(c^2 + v^2)^{1/2}$ gives

$$\Delta t = \frac{2}{(c^2+v^2)^{1/2}}\left(d^2 + \frac{v^2 \Delta t^2}{4}\right)^{1/2} \tag{6.6}$$

which has the solution $\Delta t = 2d/c$ as expected.

Eq. (6.5) is quite astonishing: it really does violate our intuition that time is absolute. We emphasise that this effect has nothing to do with the fact that we have considered light bouncing between two mirrors. We used light because it allows us to make use of Einstein's 2nd postulate. If we had used a bouncing ball then we would have become stuck when we had to figure out the speed of the ball in S because we are not entitled to assume that velocities add in the classical manner. When we have a little more knowledge and know how velocities add we will be able to return to the bouncing ball and we shall conclude that time is dilated exactly as for the light-clock. Clearly this must be the case for we are talking about the time interval between actual events.

The fact that time is actually different from our intuitive perception of it is no problem for physics, no matter how odd it may seem to us. There is a lesson to be learnt here. Namely, we should not expect our intuition based upon everday experiences to necessarily hold true in unfamiliar circumstances. In relativity theory, the unfamiliar circumstance is when objects are travelling close to the speed of light. The lesson also applies when tackling quantum theory. In this case common sense breaks down when we explore systems on very small length scales.

Example 6.1.1 *Muons are elementary particles rather like electrons but 207 times heavier. Unlike electrons, muons are unstable and they decay to an electron and a pair of neutrinos with a characteristic lifetime. For a muon at rest, this lifetime is* 2.2 μs.

Muons are created when cosmic rays impact upon the Earth's atmosphere at an altitude of 20 km and are observed to reach the Earth's surface travelling at close to the speed of light. (a) Use classical theory to estimate how far a typical muon would travel before it decays (assume the muon is travelling at the speed of light). (b) Now use time dilation to explain why the muons are able to travel the full 20 km without decaying.

Solution 6.1.1 *(a) Muons travelling at speed c will (on average) travel, according to classical thinking, a distance $c\Delta t_0$ before decaying where $\Delta t_0 = 2.2$ μs. Putting the numbers in gives a distance of just 660 m.*

(b) Let us suppose that the muon is travelling at a speed u towards the Earth. In the muon's rest frame its lifetime is a mere $\Delta t_0 = 2.2$ μs but from the point of view of an observer on Earth this lifetime is dilated to $\Delta t = \gamma \Delta t_0$. If γ is sufficiently large it is therefore possible that the muon could travel the 20 km and reach the Earth's surface. We can determine how large u must be using

$$\gamma \Delta t_0 > \frac{20\ \text{km}}{u}. \tag{6.7}$$

Since $\gamma = (1 - u^2/c^2)^{-1/2}$ we can solve this equation for $u = 0.999c$. Today, the lifetime of the muon has been measured as a function of its speed and it is found to be in excellent agreement with the prediction of time dilation.

Before leaving our discussion of time dilation we pause to consider the situation illustrated in Figure 6.3. Figure 6.3(a) shows our two frames S and S' moving relative to each other as shown. Time dilation says that, according to an observer at rest in S, clocks in S' run slow, i.e. that $\Delta t = \gamma \Delta t'$. This really does mean that all clocks run slow and so according to S an observer in S' would age more slowly. Now consider Figure 6.3(b). It represents exactly the same situation as Figure 6.3(a) since one can either think of S' moving relative to S or vice versa. Now an observer in S' will conclude that clocks in S run slow, i.e. that $\Delta t' = \gamma \Delta t$ and so from their perspective an observer at rest in S would age more slowly. At first glance these two conclusions seem to contradict each other but they do not since the observers are measuring intervals of time between different pairs of events: the observer in S is using clocks at rest in S whereas the observer in

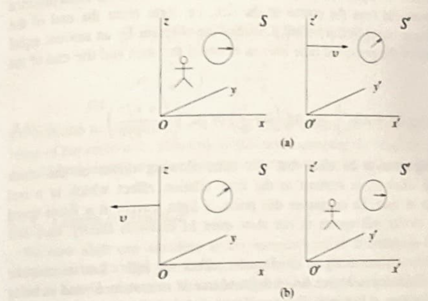

Figure 6.3 Two observers each conclude that the other is ageing more slowly than themselves. This is not a contradiction.

S' is using clocks at rest in S'. Thus it is the case that each concludes that the other is aging more slowly. Reflecting upon Einstein's 1st postulate we can see that this symmetrical situation must be correct for otherwise one could distinguish between the two inertial frames. Of course if the two observers were to meet up and compare notes then at least one of them must have undergone an acceleration. This would break the symmetry between the two and leads to the fascinating possibility that one of the observers would be genuinely older than the other upon meeting (see Section 14.1.1).

We have been very careful to explain what we mean by measurements of time and have stressed that they have nothing to do with seeing events with our eyes. Nevertheless, people do see things and it is interesting to ask how our perception of things changes in Special Relativity. Referring to Figure 5.1 we could imagine an observer situated at the origin O who is watching a clock speed away from them. We suppose that the clock is at rest at the origin O' in S'. If one tick of the clock takes a time $\Delta t'$ in S' what is the corresponding interval of time seen by the observer in S? The key word here is 'see'. Observations of events as we have hitherto been discussing them have referred explicitly to a process which does not depend upon the observer actually watching the event nor on where the observer is located when the event takes place. In contrast, the act of seeing does depend upon things like how far the observer is away from the things they are watching and the quality of the eyesight of the person doing the seeing. That distance is important when watching a moving clock becomes apparent once one appreciates that the clock is becoming ever further away and as a result light takes longer and longer to reach the observer. With this in mind, we can tackle the question in hand and attempt to work out the time interval Δt_{see} perceived by our observer at the origin O. According to all observers in S, including our observer standing at the origin, the time of one tick of the clock is given by the time dilation formula, i.e. $\Delta t = \gamma \Delta t'$. However this is not what we want. The time interval Δt_{see} is longer than Δt by an amount equal to the time it takes for light to travel the extra distance the clock has moved over the course of the tick, i.e. light from the end of the clock's tick has to travel further before it reaches the observer by an amount equal to $v\Delta t$. Therefore the perceived time interval between the start and the end of the tick is

$$\Delta t_{see} = \gamma \Delta t' + \gamma \Delta t' \frac{v}{c} = \gamma \Delta t' \left(1 + \frac{v}{c}\right) = \Delta t' \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}. \tag{6.8}$$

It is very important to be clear that this extra slowing down of the clock is an 'optical illusion', in contrast to the time dilation effect which is a real slowing down of time. To emphasise this point, if light travels at a finite speed then moving clocks will appear to run slow even in classical theory such that $\Delta t_{see} = \Delta t'(1 + v/c)$.

Eq. (6.8) leads us on nicely to the Doppler effect for light. Let us consider situation illustrated in Figure 6.4. A light source is at rest in S' and is being tched by someone at rest in S. The time interval $\Delta t'$ could just as well be the ie between the emission of successive peaks in a light wave, i.e. the frequency the wave is $f' = 1/\Delta t'$. The person watching the light source will instead see

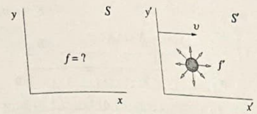

Figure 6.4 A light source of frequency f' at rest in S'.

a frequency $f = 1/\Delta t$. The two frequencies are related using Eq. (6.8):

$$f = f' \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2}. \tag{6.9}$$

This is the result in the case that the light source is moving away from the observer, in which case Eq. (6.9) tells us that $f < f'$ and so the light appears shifted to shorter frequencies, i.e. it is 'red-shifted'. If the source is moving towards the observer we should reverse the sign of v in Eq. (6.9) and therefore conclude at $f > f'$, i.e. the light is now 'blue-shifted'.

Example 6.1.2 *How fast must the driver of a car be travelling towards a red traffic light ($\lambda = 675$ nm) in order for the light to appear amber ($\lambda = 575$ nm)?*

Solution 6.1.2 *In the rest frame of the car, the traffic light is moving towards them at a speed u. Our task is to determine u given the change in wavelength. We can convert wavelengths to frequencies using $c = f\lambda$ and then use Eq. (6.9) to solve for u. Because the source is moving towards the car we should use Eq. (6.9) with $v = -u$ and so*

$$\frac{c}{575 \times 10^{-9} m} = \frac{c}{675 \times 10^{-9} m} \left(\frac{1 + u/c}{1 - u/c}\right)^{1/2},$$

$$\Rightarrow \left(\frac{675}{575}\right)^2 = \frac{1 + \beta}{1 - \beta}.$$

The solution to which is $\beta = u/c = 0.159$. It is often sensible to express speeds in terms of the ratio u/c, although in this case expressing the result as a speed of just over 13 km/s makes it clear that this effect is never going to impress a court of law.

6.1.2 Length contraction

We now shift our attention to the measurement of distances in different inertial frames and to the phenomenon known as length contraction. Light bouncing between mirrors can also be used to determine distances by accurately measuring the time it takes for light to travel between the mirrors. Let us imagine a ruler of length L_0 when measured in its rest frame. Now we ask what is the length L of the ruler when it is moving? Figure 6.5 shows a ruler moving with a speed v relative to

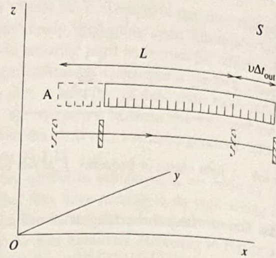

Figure 6.5 Measuring the length of a moving ruler.

S. To measure the length of the ruler we shall mount a light-clock of equal length next to it, as shown. The light-clock moves with the ruler. The light starts out from one end of the ruler and reflects from a mirror located at the opposite end of the ruler. Our strategy will be to determine the time taken for the roundtrip directly in S and equate this to the time dilation result. As a result of time dilation, the roundtrip time in S is related to the roundtrip time in the rest frame of the ruler Δt_0 by

$$\Delta t = \gamma \Delta t_0 = \gamma \frac{2L_0}{c}. \tag{6.10}$$

We shall now endeavour to determine this time interval by considering the journey of the light from the viewpoint of S. According to an observer in S, the total time is

$$\Delta t = \Delta t_{out} + \Delta t_{in}, \tag{6.11}$$

where Δt_{out} is the time taken for the light to travel on its outward journey, i.e. from A to B, and Δt_{in} is the time taken on the return journey. The figure shows explicitly the two positions of the ruler when the light starts its journey (dashed line) and when the light reaches the opposite end of the ruler (solid line). In order not to clutter the picture we have not shown the third position of the ruler, i.e. when the light finally returns back to its starting point. Since Einstein's 2nd postulate tells us the speed of light according to S, we can write

$$c\Delta t_{out} = L + v\Delta t_{out}$$

$$\Rightarrow \Delta t_{out} = \frac{L}{c - v}. \tag{6.12}$$

Each side of the first of these equations is equal to the total distance travelled by the light on its outward journey (according to S) and it takes into account the fact that the light has to travel a little further than the length of the ruler L as a result of the ruler's motion. Similarly for the return leg the light has to travel a shorter

distance than L, i.e.

$$c\Delta t_{in} = L - v\Delta t_{in}$$

$$\Rightarrow \Delta t_{in} = \frac{L}{c + v}. \tag{6.13}$$

Adding together Eqs. (6.12) and (6.13) and equating the result to Eq. (6.10) gives an equation relating L and L_0, i.e.

$$\frac{L}{c + v} + \frac{L}{c - v} = \gamma \frac{2L_0}{c}. \tag{6.14}$$

Solving for L gives

$$L = \frac{L_0}{\gamma}. \tag{6.15}$$

Again a remarkable result; for the length of the ruler is smaller when it is in motion than when it is at rest.

We could have anticipated the length contraction result knowing only the time dilation result. The argument goes as follows. Let us consider again the muons created in the upper atmosphere which we discussed in Example 6.1.1. From the viewpoint of a muon, it still lives for 2.2 μs yet has travelled all the way to the Earth's surface. However this is not such an impossible task as it would be in classical theory for the 20 km is reduced by a factor of γ. It has to be exactly the same factor of γ as before because we know that muons created at an altitude of 20 km on average just reach the Earth before decaying if they have a speed of $0.999c$ and from the viewpoint of such a muon the Earth moves towards it at that speed.

Example 6.1.3 *A spaceship flies past the Earth at a speed of $0.990c$. A crew member on the ship measures its length to be 400 m. How long is the ship as measured by an observer on Earth?*

Solution 6.1.3 *This is a straightforward application of the length contraction result expressed in Eq. (6.15) with $L_0 = 400$ m. Hence*

$$\gamma = \frac{1}{\sqrt{1 - 0.990^2}} = 7.09 \tag{6.16}$$

and so $L = 400/7.09 = 56.4$ m. Perhaps the most common misuse of the length contraction formula is to confuse L and L_0.

6.1.3 Simultaneity

Classical physics, with its absolute time, has an unambiguous notion of what it means to say two events are simultaneous. However, since time is more subjective in Special Relativity, having meaning only within the context of a specified inertial frame, it may not be suprising to hear that two events that are simultaneous in one inertial frame will not in general be simultaneous in another inertial frame. Moreover, according to one observer event A may precede event B but according to a second observer event B might occur first. This last statement sounds particularly dangerous for it suggests problems with causality. Surely everyone must agree that a person must be born before they die? And indeed they must. It is a remarkable feature of Special Relativity that although the time ordering of events can be a matter for debate this is only the case for causally disconnected events, i.e. events which cannot influence each other. We shall return to this interesting discussion in Part IV. For now we content ourselves with a thought experiment which illustrates the breakdown of simultaneity.

Consider a train travelling along at a speed u relative to the platform. An observer is standing in the middle of the train. Suppose that a flashlight is attached to each end of the train and that the flashlights flash on for a brief instant. If the observer receives the light from each flashlight at the same time then she will conclude that the flashes occurred simultaneously, for the light from each flashlight had to travel the same distance (half the length of the train) at the same speed. Now consider a second observer standing on the platform watching proceedings. They must observe that our first observer does indeed receive the light from either end of the train at a particular instant in time. However, from their viewpoint the light from the front of the train has less distance to travel than the light from the rear of the train since the observer on the train is moving towards the point of emission at the front of the train and away from the point of emission at the rear of the train. None of what has been said so far is controversial; it holds in classical theory too. Here comes the difference. As a result of the 2nd postulate, the observer on the platform still sees each pulse of light travel at the same speed c. Now since both pulses arrive at the centre of the train at the same time, and the pulse from the front had less distance to travel, it follows that it must have been emitted later than the light from the rear of the train. Classical physics avoids this conclusion because although the light from the front has less distance to travel it is travelling more slowly (its speed is $c - u$) than the light from the rear (its speed is $c + u$) and the reduction in speed compensates the reduction in distance. You might like to check that this compensation is exact and that both observers agree that the pulses were emitted at the same time according to classical physics.

6.2 LORENTZ TRANSFORMATIONS

In Section 5.1 we derived the Galilean transformation equations which relate the co-ordinates of an event in one inertial frame to the co-ordinates in a second inertial frame. For their derivation we relied upon the idea of absolute time and, as the last section showed, this is a flawed concept in Special Relativity. We must therefore seek new equations to replace the Galilean transformations. These new equations are the so-called Lorentz transformations.

To derive the Lorentz transformations we shall follow the methods of Section 5.1. We shall define our two inertial frames S and S' exactly as before, and as illustrated in Figure 5.1, i.e. S' is moving along the positive x axis at a speed v relative to S. Since the motion is parallel to the x and x' axes it follows that

$$y' = y \tag{6.17}$$

$$z' = z \tag{6.18}$$

as before. Recall that we want to express the co-ordinates in S' in terms of those measured in S. Again in order for the 1st postulate to remain valid the transformations must be of the form

$$x' = ax + bt, \tag{6.19a}$$

$$t' = dx + et. \tag{6.19b}$$

Notice that we have not assumed that there exists a unique time variable, i.e. we allow for $t' \neq t$. Our goal is to solve for the coefficients a, b, d and e. As with the derivation of the Galilean transforms we require that the origin O' (i.e. the point $x' = 0$) move along the x-axis according to $x = vt$. Substituting this information into Eq. (6.19a) yields

$$-b/a = v. \tag{6.20}$$

Similarly we require that the origin O move along the line $x' = -vt'$. From Eqs. (6.19) the point $x = 0$ satisfies $x' = bt$ and $t' = et$ such that $x' = -vt'$ implies that

$$-b/e = v. \tag{6.21}$$

Eqs. (6.20) and (6.21) imply that $e = a$ and $b = -av$. Substituting these into Eqs. (6.19) gives

$$\begin{aligned} x' &= ax - avt, \\ t' &= dx + at. \end{aligned} \tag{6.22}$$

We have two unknowns, a and d, remaining and have two postulates to implement. Let us first implement the 2nd postulate. We shall do this by considering a pulse of light emitted at the origins O and O' when they are coincident, i.e. when $t = t' = 0$. We know that this pulse must travel outwards along the x and x' axes such that it satisfies $x = ct$ and $x' = ct'$, i.e. it travels out at the same speed c in both frames. These two equations must be simultaneous solutions to Eqs. (6.22) and so we require that

$$\begin{aligned} ct' &= act - avt, \\ t' &= dct + at. \end{aligned} \tag{6.23}$$

From which it follows directly that

$$d = -\frac{av}{c^2}. \tag{6.24}$$

It only remains to determine the value of a. Let us summarise progress so far. We have reduced Eqs. (6.19a) and (6.19b) to

$$x' = a(x - vt), \tag{6.25a}$$

$$t' = a\left(t - \frac{vx}{c^2}\right). \tag{6.25b}$$