Ultrashort Laser Pulse Phenomena

Fundamentals, Techniques and Applications on a Femtosecond Time Scale

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Contents

Chapter 1

Fundamentals

1.1 Characteristics of femtosecond light pulses

Femtosecond light pulses are electromagnetic wave packets and as such are fully described by the time and space dependent electric field. In the frame of a semiclassical treatment the propagation of such fields and the interaction with matter are governed by Maxwell's equations with the material response given by a macroscopic polarization. In this first chapter we will summarize the essential notations and definitions used throughout the book. The pulse is characterized by measurable quantities which can be directly related to the electric field. A complex representation of the field amplitude is particularly convenient in dealing with propagation problems of electromagnetic pulses. The next section expands on the choice of field representation.

1.1.1 Complex representation of the electric field

Let us consider first the temporal dependence of the electric field neglecting its spatial and polarization dependence, i.e., $\mathbf{E}(x, y, z, t) = E(t)$. A complete description can be given either in the time or the frequency domain. Even though the measured quantities are real, it is generally more convenient to use complex representation. For this reason, starting with the real $E(t)$, one defines the complex spectrum of the field strength $\tilde{E}(\Omega)$, through the complex Fourier transform ($\mathcal{F}$):

$$\tilde{E}(\Omega) = \mathcal{F}\{E(t)\} = \int_{-\infty}^{\infty} E(t)e^{-i\Omega t}dt = |\tilde{E}(\Omega)|e^{i\Phi(\Omega)} \tag{1.1}$$

In the definition (1.1), $|\tilde{E}(\Omega)|$ denotes the spectral amplitude and $\Phi(\Omega)$ is the spectral phase. Here and in what follows, complex quantities related to the field are typically written with a tilde.

Since $E(t)$ is a real function, $\tilde{E}(\Omega) = \tilde{E}^*(-\Omega)$ holds. Given $\tilde{E}(\Omega)$, the time dependent electric field is obtained through the inverse Fourier transform ($\mathcal{F}^{-1}$):

$$E(t) = \mathcal{F}^{-1}\left\{\tilde{E}(\Omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\Omega)e^{i\Omega t}d\Omega \tag{1.2}$$

The physical meaning of this Fourier transform is that a pulse can be created by adding a number of waves of different frequency. Figure 1.1 sketches an ultrashort pulse created by adding continuous waves (cw). The waves are shown to be in phase at the time $t = 0$, and add constructively at that point, while destructive interference defines the temporal extension of the pulse. A single isolated pulse in time domain is constructed if the frequency difference between two successive waves is infinitesimal. In the example shown in Fig. 1.1, the frequencies are chosen to be spaced at equal frequency interval $\Delta\omega$, which implies that the same destructive interference takes place at equal time intervals $2\pi/\Delta\omega$. In this picture, the frequency spectrum is composed of a finite number of $\delta-$functions, to which correspond an infinite number of pulses in the time domain.

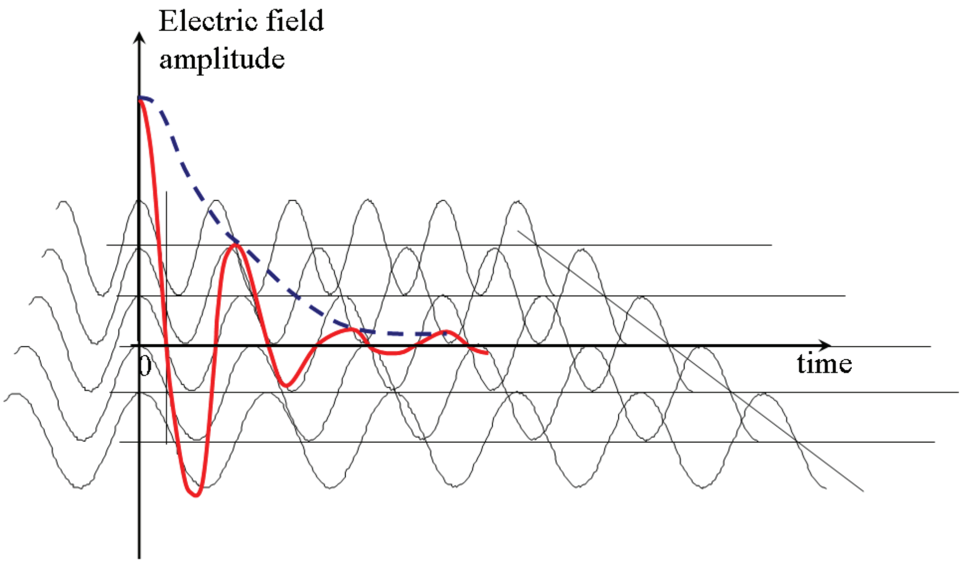

Figure 1.1: Representation of a pulse as a series of cosine waves equally spaced in frequency.

For practical reasons it may not be convenient to use functions which are non-zero for negative frequencies, as needed in the evaluation of Eq. (1.2). Frequently a complex representation of the electric field, also in the time domain, is desired. Both aspects can be satisfied by introducing a complex electric field as

$$\tilde{E}^+(t) = \frac{1}{2\pi}\int_0^\infty \tilde{E}(\Omega)e^{i\Omega t}d\Omega \tag{1.3}$$

and a corresponding spectral field strength that contains only positive frequencies:

$$\tilde{E}^+(\Omega) = |\tilde{E}(\Omega)|e^{i\Phi(\Omega)} = \begin{cases} \tilde{E}(\Omega) & \text{for} \quad \Omega \geq 0 \\ 0 & \text{for} \quad \Omega < 0 \end{cases} \tag{1.4}$$

$\tilde{E}^+(t)$ and $\tilde{E}^+(\Omega)$ are related to each other through the complex Fourier transform defined in Eq. (1.1) and Eq. (1.2), i.e.

$$\tilde{E}^+(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{E}^+(\Omega)e^{i\Omega t}d\Omega \tag{1.5}$$

and

$$\tilde{E}^+(\Omega) = \int_{-\infty}^\infty \tilde{E}^+(t)e^{-i\Omega t}dt. \tag{1.6}$$

The real physical electric field $E(t)$ and its complex Fourier transform can be expressed in terms of the quantities derived in Eq. (1.5) and Eq. (1.6) and the corresponding quantities $\tilde{E}^-(t)$, $\tilde{E}^-(\Omega)$ for the negative frequencies. These quantities relate to the real electric field:

$$E(t) = \tilde{E}^+(t) + \tilde{E}^-(t) \tag{1.7}$$

and its complex Fourier transform:

$$\tilde{E}(\Omega) = \tilde{E}^+(\Omega) + \tilde{E}^-(\Omega) \tag{1.8}$$

It can be shown that $\tilde{E}^+(t)$ can also be calculated through analytic continuation of $E(t)$

$$\tilde{E}^+(t) = E(t) + iE'(t) \tag{1.9}$$

where $E'(t)$ and $E(t)$ are Hilbert transforms of each other. In this sense $\tilde{E}^+(t)$ can be considered as the complex analytical correspondent of the real function $E(t)$. The complex electric field $\tilde{E}^+(t)$ is usually represented by a product of an amplitude function and a phase term:

$$\tilde{E}^+(t) = \frac{1}{2}\mathcal{E}(t)e^{i\Gamma(t)} \tag{1.10}$$

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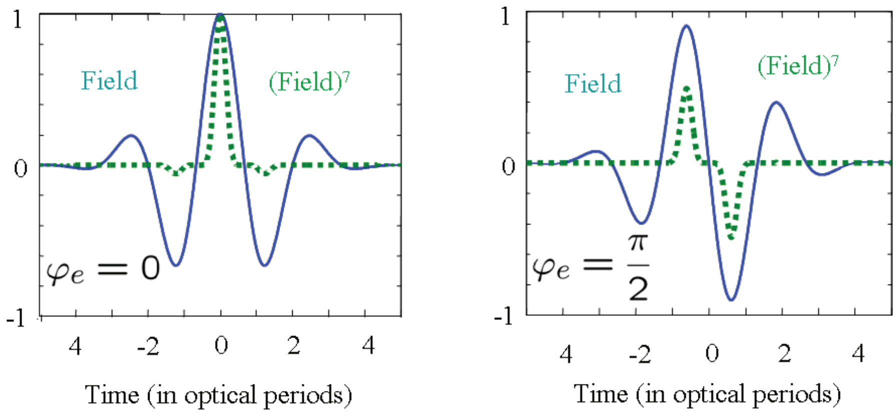

Figure 1.2: Electric field of two extremely short pulses, $E(t) = \exp[-2\ln 2(t/\tau_p)^2]\cos(\omega_\ell t + \varphi_0)$ with $\varphi_0 = 0$ (solid line) and $\varphi_0 = \pi/2$ (dashed line). Both pulses have the same envelope (dotted line). The full width of half maximum of the intensity envelope, τ_p, was chosen as $\tau_p = \pi/\omega_\ell$.

In most practical cases of interest here the spectral amplitude will be centered around a mean frequency ω_ℓ and will have appreciable values only in a frequency interval $\Delta\omega$ small compared to ω_ℓ. In the time domain this suggests the convenience of introducing a carrier frequency ω_ℓ and of writing $\tilde{E}^+(t)$ as:

$$\boxed{\tilde{E}^+(t) = \frac{1}{2}\mathcal{E}(t)e^{i\varphi_e}e^{i\varphi(t)}e^{i\omega_\ell t} = \frac{1}{2}\tilde{\mathcal{E}}(t)e^{i\omega_\ell t}} \tag{1.11}$$

where $\varphi(t)$ is the time dependent phase, $\tilde{\mathcal{E}}(t)$ is called the complex field envelope and $\mathcal{E}(t)$ the real field envelope, respectively. The constant phase term $e^{i\varphi_e}$ is most often of no relevance, and can be neglected. There are however particular circumstances pertaining to very short pulses where the outcome of the pulse interaction with matter depends on φ_e, often referred to as "carrier to envelope phase" (CEP). The measurement and control of φ_0 can therefore be quite important. Figure 1.2 shows the electric field of two pulses with identical $\mathcal{E}(t)$ but different CEP $\varphi_e = 0$ (left) and $\varphi_e = \pi/2$ (right). It is obvious that the difference can be important in the case of highly nonlinear processes, such as for instance a seven's harmonic generation creating a field proportional to the seventh power of the original field (dotted green lines).

The electric field can formally be represented in a form similar to Eq. (1.11), as illustrated by Fig. 1.2, but the mathematical entity does not always correspond to a physically possible propagating ultrashort pulses. Since the laser pulse represents a propagating electromagnetic wave packet the dc component of its spectrum vanishes. Hence the time integral over the electric field is zero.

$$\int_{-\infty}^{\infty} E(t)dt = \int_{-\infty}^{\infty} E(t)e^{-i(\Omega=0)t}dt = \mathcal{F}\left\{E(t)\right\}_{\Omega=0} = 0. \tag{1.12}$$

This not the case of the pulse with null CEP ($\varphi_e = 0$) and even less for its seventh harmonic. The convenience of representing pulse envelopes by a Gaussian or Lorentzian or secant hyperbolic envelope fails to be physical for few cycle pulses. This is illustrated in Fig. 1.3. The Fourier transform of a pulse with real electric field $\mathcal{E}(t)\cos(\omega_\ell t)$ can be constructed by shifting by $\pm$ the carrier frequency the Fourier transform of the envelope. Since the spectrum of a Gaussian has an infinite extension, the two shifted spectra will overlap at zero frequency, a non physical situation. A pulse of a few optical cycles does exist, but its representation should start with a real spectrum that has no component near zero frequency. We will discuss the carrier to

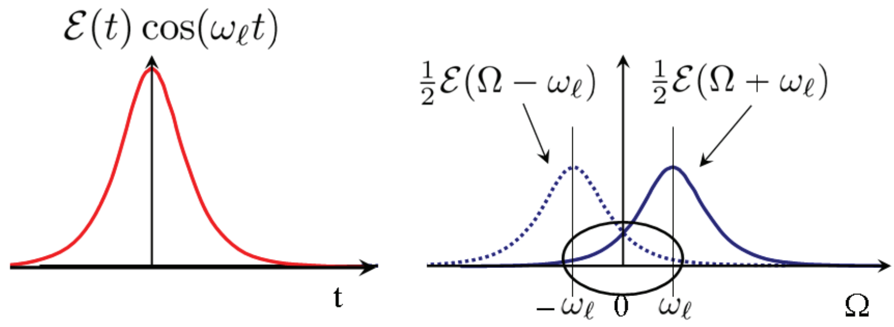

Figure 1.3: A typical pulse representation by, for instance, a Gaussian envelope at a carrier frequency ω_ℓ. The Fourier transform is constructed by shifting the Fourier transform of the envelope by $\pm\omega_\ell$, resulting in un-physical components at and near zero frequency.

envelope phase in more detail in Chapters **??** and **??**.

While the description of the field given by Eqs. (1.9) through (1.11) is quite general, the usefulness of the concept of an envelope and carrier frequency as defined in Eq. (1.11) is limited to the cases where the bandwidth is only a small fraction of the carrier frequency:

$$\frac{\Delta\omega}{\omega_\ell} \ll 1 \tag{1.13}$$

For inequality (1.13) to be satisfied, the temporal variation of $\mathcal{E}(t)$ and $\varphi(t)$ within an optical cycle $T = 2\pi/\omega_\ell$ ($T \approx 2$ fs for visible radiation) has to be small. The corresponding requirement for the complex envelope $\tilde{\mathcal{E}}(t)$ is

$$\left|\frac{d}{dt}\tilde{\mathcal{E}}(t)\right| \ll \omega_\ell \left|\tilde{\mathcal{E}}(t)\right| \tag{1.14}$$

Keeping in mind that today the shortest light pulses contain only a few optical cycles, one has to carefully check whether a slowly varying envelope and phase can describe the pulse behavior satisfactorily. If they do, the theoretical description of pulse propagation and interaction with matter can be greatly simplified by applying the slowly varying envelope approximation (SVEA), as will be evident later in this chapter.

Given the spectral description of a signal, $\tilde{E}^+(\Omega)$, the complex envelope $\tilde{\mathcal{E}}(t)$ is simply the inverse transform of the translated spectral field:

$$\tilde{\mathcal{E}}(t) = \mathcal{E}(t)e^{i\varphi(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\tilde{E}^+(\Omega + \omega_\ell)e^{i\Omega t}d\Omega; \tag{1.15}$$

where the modulus $\mathcal{E}(t)$ in Eq. (1.15) represents the real envelope. The optimum "translation" in the spectral domain ω_ℓ is the one that gives the envelope $\tilde{\mathcal{E}}(t)$ with the least amount of modulation. Spectral translation of Fourier transforms is a standard technique to reconstruct the envelope of interference patterns, and is used in Chapter **??** on diagnostic techniques. The Fourier transform of the complex envelope $\tilde{\mathcal{E}}(t)$ is the spectral envelope function:

$$\tilde{\mathcal{E}}(\Omega) = \int_{-\infty}^{\infty} \tilde{\mathcal{E}}(t)e^{-i\Omega t}dt = 2\int_{-\infty}^{\infty} \tilde{E}^+(t)e^{-i(\Omega+\omega_\ell)t}dt. \tag{1.16}$$

The choice of ω_ℓ is such that the spectral amplitude $\tilde{\mathcal{E}}(\Omega)$ is centered about the origin $\Omega = 0$.

Let us now discuss more carefully the physical meaning of the phase function $\varphi(t)$. The choice of carrier frequency in Eq. (1.11) should be such as to minimize the variation of phase $\varphi(t)$. The first derivative of the phase factor $\Gamma(t)$ in Eq. (1.10) establishes a time dependent carrier frequency (instantaneous frequency):

$$\boxed{\omega(t) = \omega_\ell + \frac{d}{dt}\varphi(t).} \tag{1.17}$$

While Eq. (1.17) can be seen as a straightforward definition of an instantaneous frequency based on the temporal variation of the phase factor $\Gamma(t)$,

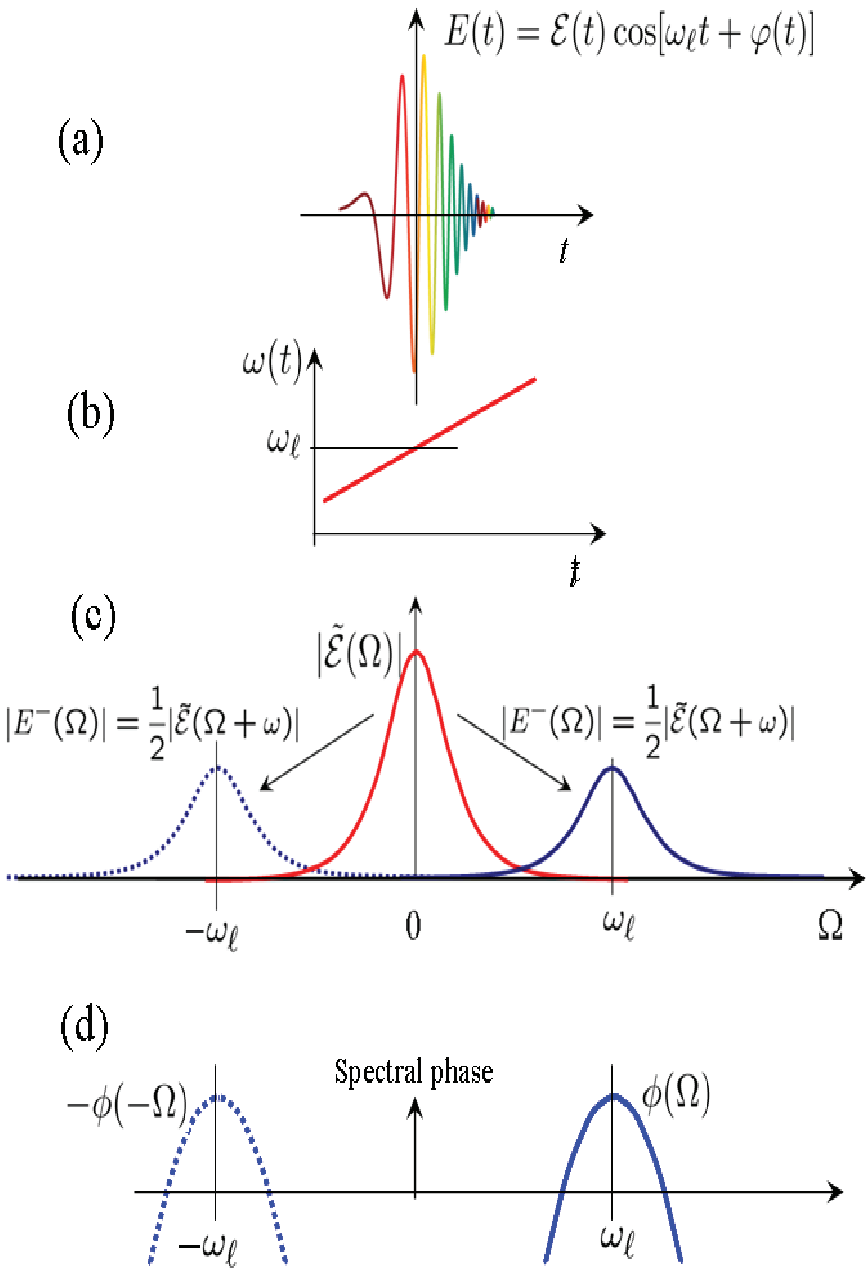

Figure 1.4: (a) Electric field, (b) time dependent carrier frequency, (c) spectral amplitude and (d) spectral phase of a linearly upchirped pulse.

we will see in Section 1.1.4 that it can be rigourously derived from the Wigner distribution. For $d\varphi/dt = b =$ const., a non-zero value of b just means a correction of the carrier frequency which is now $\omega'_\ell = \omega_\ell + b$. For $d\varphi/dt = f(t)$, the carrier frequency varies with time and the corresponding pulse is said to be frequency modulated or chirped. For $d^2\varphi/dt^2 < (>) 0$, the carrier frequency decreases (increases) along the pulse, which then is called down(up)chirped.

From Eq.(1.10) it is obvious that the decomposition of $\Gamma(t)$ into ω and $\varphi(t)$ is not unique. The most useful decomposition is one that ensures the smallest $d\varphi/dt$ during the intense portion of the pulse. A common practice is to identify ω_ℓ with the carrier frequency at the pulse peak. A better definition — which is consistent in the time and frequency domains — is to use the intensity weighted *average* frequency:

$$\boxed{\langle\omega\rangle = \frac{\int_{-\infty}^{\infty}|\tilde{\mathcal{E}}(t)|^2\omega(t)dt}{\int_{-\infty}^{\infty}|\tilde{\mathcal{E}}(t)|^2dt} = \frac{\int_{-\infty}^{\infty}|\tilde{E}^+(\Omega)|^2\Omega d\Omega}{\int_{-\infty}^{\infty}|\tilde{E}^+(\Omega)|^2d\Omega}} \tag{1.18}$$

The various notations are illustrated in Fig. 1.4 where a linearly upchirped pulse is taken as an example. The temporal dependence of the real electric field is sketched in the top part of Fig 1.4. A complex representation in the time domain is illustrated with the amplitude and instantaneous frequency of the field. The positive and negative frequency components of the Fourier transform are shown in amplitude and phase in the bottom part of the figure.

1.1.2 Power, energy, and related quantities

Let us imagine the practical situation in which the pulse propagates as a beam with cross section A, and with $E(t)$ as the relevant component of the electric field. The (instantaneous) pulse power (in Watt) in a dispersionless material of refractive index n can be derived from the Poynting theorem of electrodynamics [1] and is given by

$$\mathcal{P}(t) = \epsilon_0 cn\int_A dS\frac{1}{T}\int_{t-T/2}^{t+T/2}E^2(t')dt' \tag{1.19}$$

where c is the velocity of light in vacuum, ϵ_0 is the dielectric permittivity and $\int_A dS$ stands for integration over the beam cross section. The power can be measured by a detector (photodiode, photomultiplier etc.) which integrates over the beam cross section. The temporal response of this device must be short as compared to the speed of variations of the field envelope to be measured. The temporal averaging is performed over one optical period $T = 2\pi/\omega_\ell$. Note that the instantaneous power as introduced in Eq. (1.19) is then just a convenient theoretical quantity. In a practical measurement T has to be replaced by the actual response time τ_R of the detector. Therefore, even with the fastest detectors available today ($\tau_R \approx 10^{-13} - 10^{-12}$s), details of the envelope of fs light pulses can not be resolved directly.

A temporal integration of the power yields the energy $\mathcal{W}$ (in Joules):

$$\mathcal{W} = \int_{-\infty}^{\infty} \mathcal{P}(t')dt' \tag{1.20}$$

where the upper and lower integration limits essentially mean "before" and "after" the pulse under investigation.

The corresponding quantity per unit area is the intensity (W/cm^2):

$$\begin{aligned} I(t) &= \epsilon_0 cn \frac{1}{T}\int_{t-T/2}^{t+T/2} E^2(t')dt' \\ &= \frac{1}{2}\epsilon_0 cn\mathcal{E}^2(t) = 2\epsilon_0 cn\tilde{E}^+(t)\tilde{E}^-(t) = \frac{1}{2}\epsilon_0 cn\tilde{\mathcal{E}}(t)\tilde{\mathcal{E}}^*(t) \end{aligned} \tag{1.21}$$

and the energy density per unit area (J/cm^2):

$$W = \int_{-\infty}^{\infty} I(t')dt' \tag{1.22}$$

Sometimes it is convenient to use quantities which are related to photon numbers, such as the photon flux $\mathcal{F}$ (photons/s) or the photon flux density F (photons/s/cm^2):

$$\mathcal{F}(t) = \frac{\mathcal{P}(t)}{\hbar\omega_\ell} \quad \text{and} \quad F(t) = \frac{I(t)}{\hbar\omega_\ell} \tag{1.23}$$

where $\hbar\omega_\ell$ is the energy of one photon at the carrier frequency.

The spectral properties of the light are typically obtained by measuring the intensity of the field, without any time resolution, at the output of a spectrometer. The quantity, called spectral intensity, that is measured is:

$$S(\Omega) = \mid \eta(\Omega)\tilde{E}^+(\Omega) \mid^2 \tag{1.24}$$

where η is a scaling factor which accounts for losses, geometrical influences, and the finite resolution of the spectrometer. Assuming an ideal spectrometer, $|\eta|^2$ can be determined from the requirement of energy conservation:

$$|\eta|^2 \int_{-\infty}^{\infty} \mid \tilde{E}^+(\Omega) \mid^2 d\Omega = 2\epsilon_0 cn \int_{-\infty}^{\infty} \tilde{E}^+(t)\tilde{E}^-(t)dt \tag{1.25}$$

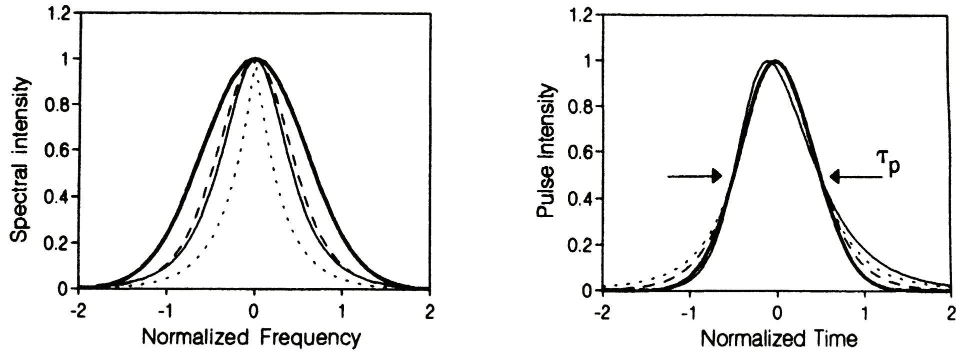

Figure 1.5: Temporal pulse profiles and the corresponding spectra (normalized).

———— Gaussian pulse	$\mathcal{E}(t)$	$\propto$	$\exp[-1.385(t/\tau_p)^2]$
- - - - - - - sech - pulse	$\mathcal{E}(t)$	$\propto$	$\text{sech}[1.763(t/\tau_p)]$
· · · · · · · · · Lorentzian pulse	$\mathcal{E}(t)$	$\propto$	$[1+1.656(t/\tau_p)^2]^{-1}$
——— asymm. sech pulse	$\mathcal{E}(t)$	$\propto$	$[\exp(t/\tau_p)+\exp(-3t/\tau_p)]^{-1}$

and Parseval's theorem [2]:

$$\int_{-\infty}^{\infty} |\tilde{E}^+(t)|^2 dt = \frac{1}{2\pi}\int_0^{\infty} \mid \tilde{E}^+(\Omega) \mid^2 d\Omega \tag{1.26}$$

from which follows $|\eta|^2 = \epsilon_0 cn/\pi$. The complete expression for the spectral intensity [from Eq. (1.24)] is thus:

$$S(\Omega) = \frac{\epsilon_0 cn}{4\pi}\left|\tilde{\mathcal{E}}(\Omega - \omega_\ell)\right|^2 . \tag{1.27}$$

Figure 1.5 gives examples of typical pulse shapes and the corresponding spectra.

The complex quantity $\tilde{E}^+$ will be used most often throughout the book to describe the electric field. Therefore, to simplify notations, we will omit the superscript "+"whenever this will not cause confusion.

1.1.3 Pulse duration and spectral width

Unless specified otherwise, we define the pulse duration τ_p as the full width at half maximum (FWHM) of the intensity profile, $|\tilde{\mathcal{E}}(t)|^2$, and the spectral width $\Delta\omega_p$ as the FWHM of the spectral intensity $|\tilde{\mathcal{E}}(\Omega)|^2$. Making that statement is an obvious admission that other definitions exist. Precisely because of the difficulty of asserting the exact pulse shape, standard waveforms have been selected. The most commonly cited are the Gaussian, for which the temporal dependence of the field is:

$$\tilde{\mathcal{E}}(t) = \tilde{\mathcal{E}}_0 \exp\{-(t/\tau_G)^2\} \tag{1.28}$$

and the secant hyperbolic:

$$\tilde{\mathcal{E}}(t) = \tilde{\mathcal{E}}_0 \mathrm{sech}(t/\tau_s). \tag{1.29}$$

The parameters $\tau_G = \tau_p/\sqrt{2\ln 2}$ and $\tau_s = \tau_p/1.76$ are generally more convenient to use in theoretical calculations involving pulses with these assumed shapes than the FWHM of the intensity, τ_p.

Since the temporal and spectral characteristics of the field are related to each other through Fourier transforms, the bandwidth $\Delta\omega_p$ and pulse duration τ_p cannot vary independently of each other. There is a minimum duration-bandwidth product:

$$\Delta\omega_p\,\tau_p = 2\pi\Delta\nu_p\tau_p \geq 2\pi c_B. \tag{1.30}$$

c_B is a numerical constant on the order of 1, depending on the actual pulse shape. Some examples are shown in Table 1.1. The equality holds for pulses without frequency modulation (unchirped) which are called "bandwidth limited" or "Fourier limited". Such pulses exhibit the shortest possible duration at a given spectral width and pulse shape. We refer the reader to Section 1.1.4, for a more general discussion of the uncertainty relation between pulse and spectral width based on mean-square deviations.

The shorter the pulse duration, the more difficult it becomes to assert its detailed characteristics. In the femtosecond domain, even the simple concept of pulse duration seems to fade away in a cloud of mushrooming definitions. Part of the problem is that it is difficult to determine the exact pulse shape. For single pulses, the typical representative function that is readily accessible to the experimentalist is the intensity autocorrelation:

$$A_{\mathrm{int}}(\tau) = \int_{-\infty}^{\infty} I(t)I(t-\tau)dt \tag{1.31}$$

The Fourier transform of the correlation (1.31) is the real function:

$$A_{\mathrm{int}}(\Omega) = \tilde{\mathcal{I}}(\Omega)\tilde{\mathcal{I}}^*(\Omega) \tag{1.32}$$

where the notation $\tilde{\mathcal{I}}(\Omega)$ is the Fourier transform of the function $I(t)$, which should not be confused with the spectral intensity $S(\Omega)$. The fact that

Shape	Intensity profile $I(t)$	τ_p FWHM	Spectral profile $S(\Omega)$	$\Delta\omega_p$ FWHM	c_B	$\langle\tau_p\rangle\langle\Delta\Omega_p\rangle$ MSQ
Gauss	$e^{-2(t/\tau_G)^2}$	$1.177\tau_G$	$e^{-\frac{(\Omega\tau_G)^2}{2}}$	$2.355/\tau_G$	0.441	0.5
sech	$\text{sech}^2(t/\tau_s)$	$1.763\tau_s$	$\text{sech}^2\frac{\pi\Omega\tau_s}{2}$	$1.122/\tau_s$	0.315	0.525
Lorentz	$[1+(t/\tau_L)^2]^{-2}$	$1.287\tau_L$	$e^{-2\|\Omega\|\tau_L}$	$0.693/\tau_L$	0.142	0.7
asym. sech	$\left[e^{t/\tau_a}+e^{-3t/\tau_a}\right]^{-2}$	$1.043\tau_a$	$\text{sech}\frac{\pi\Omega\tau_a}{2}$	$1.677/\tau_a$	0.278	
square	1 for $\|t/\tau_r\|\leq 1$, 0 elsewhere	τ_r	$\text{sinc}^2(\Omega\tau_r)$	$2.78/\tau_r$	0.443	3.27

Table 1.1: Examples of standard pulse profiles. The spectral values given are for unmodulated pulses. Note that the Gaussian is the shape with the minimum product of mean square deviation (MSQ) of the intensity and spectral intensity.

the autocorrelation function $A_{\text{int}}(\tau)$ is symmetric, hence its Fourier transform is real [2], implies that little information about the pulse shape can be extracted from such a measurement. Furthermore, the intensity autocorrelation (1.31) contains no information about the pulse phase or coherence. This point is discussed in detail in Chapter **??**.

Gaussian pulses

Having introduced essential pulse characteristics, it seems convenient to discuss an example to which we can refer to in later chapters. We choose a Gaussian pulse with linear chirp. This choice is one of analytical convenience: the Gaussian shape is *not* the most commonly encountered temporal shape. The electric field is given by

$$\tilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{-(1+ia)(t/\tau_G)^2} \tag{1.33}$$

with the pulse duration

$$\tau_p = \sqrt{2\ln 2}\,\tau_G. \tag{1.34}$$

Note that with the definition (1.33) the chirp parameter a is positive for a downchirp ($d\varphi/dt = -2at/\tau_G^2$). The Fourier transform of (1.33) yields

$$\tilde{\mathcal{E}}(\Omega) = \frac{\mathcal{E}_0\sqrt{\pi}\tau_G}{\sqrt[4]{1+a^2}} \exp\left\{i\Phi - \frac{\Omega^2{\tau_G}^2}{4(1+a^2)}\right\} \tag{1.35}$$

with the spectral phase given by:

$$\phi(\Omega) = -\frac{1}{2}\arctan(a) + \frac{a{\tau_G}^2}{4(1+a^2)}\Omega^2 \tag{1.36}$$

It can be seen from Eq. (1.35) that the spectral intensity is the Gaussian:

$$S(\omega_\ell + \Omega) = \frac{|\eta|^2 \pi \mathcal{E}_0^2 \tau_G^2}{\sqrt{1+a^2}} \exp\left\{-\frac{\Omega^2 {\tau_G}^2}{2(1+a^2)}\right\} \tag{1.37}$$

with a FWHM given by:

$$\Delta\omega_p = 2\pi\Delta\nu_p = \frac{1}{\tau_G}\sqrt{8\ln 2(1+a^2)} \tag{1.38}$$

For the pulse duration-bandwidth product we find

$$\Delta\nu_p \tau_p = \frac{2\ln 2}{\pi}\sqrt{1+a^2} \tag{1.39}$$

Obviously, the occurrence of chirp ($a \neq 0$) results in additional spectral components which enlarge the spectral width and lead to a duration bandwidth product exceeding the Fourier limit ($2\ln 2/\pi \approx 0.44$) by a factor $\sqrt{1+a^2}$, consistent with Eq. (1.30). We also want to point out that the spectral phase given by Eq. (1.36) changes quadratically with frequency if the input pulse is linearly chirped. While this is exactly true for Gaussian pulses as can be seen from Eq. (1.36), it holds approximately for other pulse shapes. In the next section, we will develop a concept that allows one to discuss the pulse duration-bandwidth product from a more general point of view and independent of the actual pulse and spectral profile.

1.1.4 Wigner distribution, second order moments, uncertainty relations

Wigner distribution

The Fourier transform as defined in Section 1.1.1 is a widely used tool in beam and pulse propagation. In beam propagation, it leads directly to the far field pattern of a propagating beam (Fraunhofer approximation) of arbitrary transverse profile. Similarly, the Fourier transform leads directly to the pulse temporal profile, following propagation through a dispersive medium, as we will see at the end of this chapter. The Fourier transform gives a weighted average of the spectral components contained in a signal.

Unfortunately, the exact spatial or temporal location of these spectral components is hidden in the phase of the spectral field. There has been therefore a need for new two-dimensional representation of the waves in either the plane of space–wave vector, or time–angular frequency. Such a function was introduced by Wigner [3] and applied to quantum mechanics. The same distribution was applied to the area of signal processing by Ville [4]. Properties and applications of the Wigner distribution in Quantum Mechanics and Optics are reviewed in two recent books by Schleich [5] and Cohen [6]. A clear analysis of the close relationship between Quantum Mechanics and Optics can be found in ref. [7]. The Wigner distribution of a function $\tilde{E}(t)$ is defined by[1]:

$$\begin{aligned}\mathcal{W}_E(t,\Omega) &= \int_{-\infty}^{\infty} \tilde{E}\left(t+\frac{s}{2}\right)\tilde{E}^*\left(t-\frac{s}{2}\right)e^{-i\Omega s}ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}\left(\Omega+\frac{s}{2}\right)\tilde{E}^*\left(\Omega-\frac{s}{2}\right)e^{its}ds \end{aligned} \tag{1.40}$$

One can see that the definition is a local representation of the spectrum of the signal, since:

$$\int_{-\infty}^{\infty} \mathcal{W}_E(t,\Omega)dt = \left|\tilde{E}(\Omega)\right|^2 \tag{1.41}$$

and

$$\int_{-\infty}^{\infty} \mathcal{W}_E(t,\Omega)d\Omega = 2\pi\left|\tilde{E}(t)\right|^2 \tag{1.42}$$

The subscript E refers to the use of the instantaneous complex electric field $\tilde{E}$ in the definition of the Wigner function, rather than the electric field envelope $\tilde{\mathcal{E}} = \mathcal{E}\exp[i\omega_\ell t + i\varphi(t)]$ defined at the beginning of this chapter. There is a simple relation between the Wigner distribution $\mathcal{W}_E$ of the instantaneous field $\tilde{E}$, and the Wigner distribution $\mathcal{W}_\mathcal{E}$ of the real envelope amplitude $\mathcal{E}$:

$$\begin{aligned}\mathcal{W}_E(t,\Omega) &= \int_{-\infty}^{\infty} \mathcal{E}\left(t+\frac{s}{2}\right)e^{i[\omega_\ell(t+s/2)+\varphi(t+s/2)]} \\ &\quad\times\ \mathcal{E}^*\left(t-\frac{s}{2}\right)e^{-i[\omega_\ell(t-s/2)+\varphi(t-s/2)]}e^{-i\Omega s}ds \\ &= \int_{-\infty}^{\infty} \mathcal{E}\left(t+\frac{s}{2}\right)\mathcal{E}^*\left(t-\frac{s}{2}\right)e^{-i[\Omega-(\omega_\ell+\dot{\varphi}(t))]s}ds \\ &= \mathcal{W}_\mathcal{E}\{t,[\Omega-(\omega_\ell+\dot{\varphi})]\}. \end{aligned} \tag{1.43}$$

[1] t and Ω are conjugated variables as in Fourier transforms. The same definitions can be made in the space–wavevector domain, where the variables are then x and k.

We will drop the subscript "E" and "$\mathcal{E}$" for the Wigner function when the distinction is not essential.

The intensity and spectral intensities are directly proportional to frequency and time integrations of the Wigner function. In accordance with Eqs. (1.21) and Eq. (1.27):

$$\frac{1}{2\sqrt{\mu_0/\epsilon}}\int_{-\infty}^{\infty}\mathcal{W}_{\mathcal{E}}(t,\Omega)d\Omega = I(t) \tag{1.44}$$

$$\frac{1}{2\sqrt{\mu_0/\epsilon}}\int_{-\infty}^{\infty}\mathcal{W}_{\mathcal{E}}(t,\Omega)dt = S(\Omega). \tag{1.45}$$

Figure 1.6 shows the Wigner distribution of an unchirped Gaussian pulse ((a), left) versus a Gaussian pulse with a quadratic chirp ((b), right). The introduction of a quadratic phase modulation leads to a tilt (rotation) and flattening of the distribution. This distortion of the Wigner function results directly from the relation (1.43) applied to a Gaussian pulse. We have defined in Eq. (1.33) the phase of the linearly chirped pulse as $\varphi(t) = -at^2/\tau_G^2$. If $\mathcal{W}_{\text{unchirp}}$ is the Wigner distribution of the unchirped pulse, the linear chirp transforms that function into:

$$\mathcal{W}_{\text{chirp}} = \mathcal{W}_{\text{unchirp}}(t, \Omega - \frac{2at}{\tau_G^2}), \tag{1.46}$$

hence the tilt observed in Fig. 1.6. Mathematical tools have been developed to produce a pure rotation of the phase space (t, Ω). We refer the interested reader to the literature for details on the Wigner distribution and in particular on the fractional Fourier transform [8, 9]. It has been shown that such a rotation describes the propagation of a pulse through a medium with a quadratic dispersion (index of refraction being a quadratic function of frequency) [10].

Moments of the electric field

It is mainly history and convenience that led to the adoption of the FWHM of the pulse intensity as the quantity representative of the pulse duration. Sometimes pulse duration and spectral width defined by the FWHM values are not suitable measures. This is, for instance, the case in pulses with substructure or broad wings causing a considerable part of the energy to lie outside the range given by the FWHM. In these cases it may be preferable to use averaged values derived from the appropriate second–order moments. It appears in fact, as will be shown in examples of propagation, that the second moment of the field distribution is a better choice.

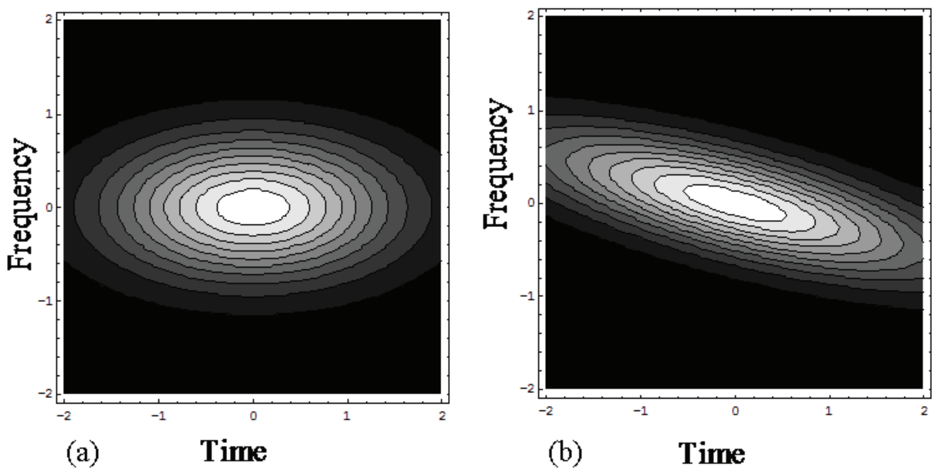

Figure 1.6: Wigner distribution for a Gaussian pulse. Left (a), the phase function $\varphi(t) = \varphi_0$ is a constant. On the right (b), Wigner distribution for a linearly chirped pulse, i.e. with a quadratic phase modulation $\varphi(t) = \alpha t^2$. The elliptical curves are lines of equal intensity. The intensity is graded from 0 (black) to the peak (white).

For the sake of generality, let us designate by $f(x)$ the field as a function of the variable x (which can be the transverse coordinate, transverse wave vector, time or frequency). The moment of order n for the quantity x with respect to intensity is defined as:

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} x^n |f(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \tag{1.47}$$

The first order moment, $\langle x \rangle$, is the "center of mass" of the intensity distribution, and is most often chosen as reference, in such a way as to have a zero value. For example, the center of the transverse distribution will be on axis, $x = 0$, or a Gaussian temporal intensity distribution $\mathcal{E}_0 \exp[-(t/\tau_G)^2]$ will be centered at $t = 0$. A good criterium for the width of a distribution is the mean square deviation (MSQ):

$$\langle \Delta x \rangle = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}. \tag{1.48}$$

The explicit expressions in the time and frequency domains are:

$$\langle \tau_p \rangle = \langle \Delta t \rangle = \left[\frac{1}{W} \int_{-\infty}^{\infty} t^2 I(t) dt - \frac{1}{W^2} \left(\int_{-\infty}^{\infty} t I(t) dt \right)^2 \right]^{\frac{1}{2}} \tag{1.49}$$

$$\langle\Delta\omega_p\rangle = \langle\Delta\Omega\rangle = \left[\frac{1}{W}\int_{-\infty}^{\infty}\Omega^2 S(\Omega)d\Omega - \frac{1}{W^2}\left(\int_{-\infty}^{\infty}\Omega S(\Omega)d\Omega\right)^2\right]^{\frac{1}{2}} \quad (1.50)$$

where $S(\Omega)$ is the spectral intensity defined in Eq. (1.24). Whenever appropriate we will assume that the first-order moments are zero, which yields $\langle\Delta x\rangle = \sqrt{\langle x^2\rangle}$.

The second moments can also be defined using the Wigner distribution [Eq. (1.40)]:

$$\langle t^2\rangle = \frac{\int\int_{-\infty}^{\infty} t^2\mathcal{W}_E(t,\Omega)dtd\Omega}{\int\int_{-\infty}^{\infty}\mathcal{W}_E(t,\Omega)dtd\Omega} = \frac{\int_{-\infty}^{\infty} t^2|\tilde{E}(t)|^2dt}{\int_{-\infty}^{\infty}|\tilde{E}(t)|^2dt} \quad (1.51)$$

$$\langle\Omega^2\rangle = \frac{\int\int_{-\infty}^{\infty}\Omega^2\mathcal{W}_E(t,\Omega)dtd\Omega}{\int\int_{-\infty}^{\infty}\mathcal{W}_E(t,\Omega)dtd\Omega} = \frac{\int_{-\infty}^{\infty}\Omega^2|\tilde{E}(\Omega)|^2d\Omega}{\int_{-\infty}^{\infty}|\tilde{E}(\Omega)|^2d\Omega} \quad (1.52)$$

While the above equations do not bring anything new, the Wigner distribution lets us define another quantity, which describes the coupling between conjugated variables:

$$\langle t,\Omega\rangle = \frac{\int\int_{-\infty}^{\infty}(t-\langle t\rangle)(\Omega-\langle\Omega\rangle)\mathcal{W}_E(t,\Omega)dtd\Omega}{\int\int_{-\infty}^{\infty}\mathcal{W}_E(t,\Omega)dtd\Omega}. \quad (1.53)$$

A non-zero $\langle t,\Omega\rangle$ implies that the center of mass of the spectral intensity evolves with time, as in Fig. 1.6. One can thus define an instantaneous frequency:

$$\omega(t) = \frac{\int_{-\infty}^{\infty}\Omega\mathcal{W}_E(t,\Omega)d\Omega}{\int_{-\infty}^{\infty}\mathcal{W}_E(t,\Omega)d\Omega}. \quad (1.54)$$

By substituting the definition of the Wigner distribution Eq. (1.40) in Eq. (1.54), it is possible to demonstrate rigourously the relation (1.17). Indeed, substituting the definition (1.43) in Eq. (1.54) leads to:

$$\begin{aligned}\omega(t) &= \frac{\int_{-\infty}^{\infty}\Omega\mathcal{W}_{\mathcal{E}}[t,\Omega-(\omega_\ell+\dot{\varphi})]d\Omega}{\int_{-\infty}^{\infty}\mathcal{W}_E(t,\Omega)d\Omega}\\ &= \frac{\int_{-\infty}^{\infty}[\Omega'+\omega_\ell+\dot{\varphi}(t)]\mathcal{W}_{\mathcal{E}}[t,\Omega']d\Omega'}{\int_{-\infty}^{\infty}\mathcal{W}_E(t,\Omega)d\Omega}\\ &= \omega_\ell+\dot{\varphi}(t),\end{aligned} \quad (1.55)$$

where we used the fact that $\int\Omega'\mathcal{W}_{\mathcal{E}}(t,\Omega')d\Omega' = 0$.

There is a well known uncertainty principle between the second moment of conjugated variables. If k is the Fourier-conjugated variable of x, it is shown in Appendix **??** that:

$$\langle x^2\rangle\langle k^2\rangle = \frac{M^4}{4} \geq \frac{1}{4}, \tag{1.56}$$

where we have defined a shape factor "M^2", which has been extensively used to describe the departure of beam profile from the "ideal Gaussian" [11]. This relation can be applied to time and frequency:

$$\langle t^2\rangle\langle \Omega^2\rangle = \frac{M^4}{4} \geq \frac{1}{4}. \tag{1.57}$$

Equality only holds for a Gaussian pulse (beam) shape free of any phase modulation, which implies that the Wigner distribution for a Gaussian shape occupies the smallest area in the time/frequency plane. It is also important to note that the uncertainty relations (1.56) and (1.57) only hold for the pulse widths defined as the mean square deviation. For a Gaussian pulses defined by its electric field $\mathcal{E}(t) = \mathcal{E}_0 \exp[-(t/\tau_G)^2]$:

$$\begin{aligned} \langle t^2\rangle &= \frac{\tau_G^2}{4} \\ \langle \Omega^2\rangle &= \frac{1}{\tau_G^2}. \end{aligned} \tag{1.58}$$

The product of the two numbers is indeed 1/4, the minimum of the inequality (1.57). while for the products of the full width at half maximum (FWHM) of the intensity and spectral intensity $c_B = \tau_p \Delta\nu_p = 0.441$. In fact, the pulse duration-bandwidth product *is not minimum* for a Gaussian pulse, as illustrated in Table 1.1, which gives the value of c_B for various pulse shapes without phase modulation. It remains that, for a given pulse shape, c_B is the smallest for pulses without frequency modulation (unchirped) which are called "bandwidth limited" or "Fourier limited". Such pulses exhibit the shortest possible duration at a given spectral width and pulse shape.

If there is a frequency variation across a pulse, its spectrum will contain additional spectral components. Consequently, the modulated pulse possesses a spectral width which is larger than the Fourier limit given by column five in Table 1.1.

Chirped pulses

A quadratic phase modulation plays an essential role in light propagation, be it in time or space. Since a spherical wavefront can be approximated by a quadratic phase ($\varphi(x) \propto x^2$, where x is the transverse dimension) near any propagation axis of interest, imparting a quadratic spatial phase modulation will lead to focusing or de-focusing of a beam. The analogue is true in time: imparting a quadratic phase modulation ($\varphi(t) \propto t^2$) will lead to pulse compression or broadening after propagation through a dispersive medium. These problems relating to pulse propagation will be discussed in several sections and chapters of this book. In this section we attempt to clarify quantitatively the relation between a quadratic chirp in the temporal or frequency space, and the corresponding broadening of the spectrum or pulse duration, respectively. The results are interchangeable from frequency to temporal space.

Let us first assume that a laser pulse, initially unchirped, propagates through a dispersive material that leaves the pulse spectrum, $|\tilde{\mathcal{E}}(\Omega)|^2$, unchanged but produces a quadratic phase modulation in the frequency domain. The pulse spectrum is centered at the average frequency $\langle\Omega\rangle = \omega_\ell$. The average frequency does not change, hence the first nonzero term in the Taylor expansion of $\phi(\Omega)$ is

$$\phi(\Omega) = \frac{1}{2}\left.\frac{d^2\phi}{d\Omega^2}\right|_0 \langle\Omega^2\rangle, \tag{1.59}$$

where $\phi(\Omega)$ determines the phase factor of $\tilde{\mathcal{E}}(\Omega)$:

$$\tilde{\mathcal{E}}(\Omega) = \mathcal{E}(\Omega)e^{i\phi(\Omega)}. \tag{1.60}$$

The first and second order moments are, according to the definitions (1.47):

$$\langle t\rangle = \frac{\int_{-\infty}^{\infty} t\tilde{\mathcal{E}}(t)\tilde{\mathcal{E}}(t)^* dt}{\int_{-\infty}^{\infty}|\tilde{\mathcal{E}}(t)|^2 dt} = \frac{\int_{-\infty}^{\infty}\frac{d\tilde{\mathcal{E}}(\Omega)}{d\Omega}\tilde{\mathcal{E}}^*(\Omega)d\Omega}{\int_{-\infty}^{\infty}|\tilde{\mathcal{E}}(\Omega)|^2 d\Omega} = \left\langle\frac{d\phi}{d\Omega}\right\rangle \tag{1.61}$$

and

$$\begin{aligned}\langle t^2\rangle &= \frac{\int_{-\infty}^{\infty} t\tilde{\mathcal{E}}(t)t\tilde{\mathcal{E}}(t)^* dt}{\int_{-\infty}^{\infty}|\tilde{\mathcal{E}}(t)|^2 dt} = \frac{\int_{-\infty}^{\infty}\left|\frac{d\tilde{\mathcal{E}}(\Omega)}{d\Omega}\right|^2 d\Omega}{\int_{-\infty}^{\infty}|\tilde{\mathcal{E}}(t)|^2 dt} \\ &= \frac{\int_{-\infty}^{\infty}\left[\frac{d\mathcal{E}(\Omega)}{d\Omega}\right]^2 d\Omega}{\int_{-\infty}^{\infty}|\tilde{\mathcal{E}}(\Omega)|^2 d\Omega} + \left\langle\left(\frac{d\phi}{d\Omega}\right)^2\right\rangle.\end{aligned} \tag{1.62}$$

It is left to a problem at the end of this chapter to derive these results. Since the initial pulse was unchirped and its spectral amplitude is not affected by propagation through the transparent medium, the first term in Eq. (1.62) represents the initial second order moment $\langle t^2 \rangle_0$. Substituting the expression for the quadratic phase Eq. (1.59) into Eq. (1.47) for the first order moment, we find from Eq. (1.62):

$$\langle t^2 \rangle = \langle t^2 \rangle_0 + \left[\left. \frac{d^2\phi}{d\Omega^2} \right|_0 \right]^2 \langle \Omega^2 \rangle. \tag{1.63}$$

The frequency chirp introduces a temporal broadening (of the second order moment) directly proportional to the square of the chirp coefficient, $\left[\left. \frac{d^2\phi}{d\Omega^2} \right|_0 \right]^2$.

Likewise we can analyze the situation where a temporal phase modulation $\varphi(t) = \left. \frac{d\varphi}{dt} \right|_0 t^2$ is impressed upon the pulse while the pulse envelope, $|\tilde{\mathcal{E}}(t)|^2$, remains unchanged. This temporal frequency modulation or chirp, characterized by the second derivative in the middle (center of mass) of the pulse, leads to a spectral broadening given by:

$$\langle \Omega^2 \rangle = \langle \Omega^2 \rangle_0 + \left[\left. \frac{d^2\varphi}{dt^2} \right|_0 \right]^2 \langle t^2 \rangle \tag{1.64}$$

where $\langle \Omega^2 \rangle_0$ refers to the spectrum of the input pulse and $\langle t^2 \rangle$ is the (constant) second-order moment of time.

Equations (1.63) and (1.64) demonstrate the advantage of using the mean square deviation to define the pulse duration and bandwidth, since it shows a simple relation between the broadening in the time or spectral domain, due to a chirp in the spectral or time domain, respectively independent of the pulse and spectral shape. For the two different situations described by Eqs. (1.63) and (1.64), we can apply the uncertainty relation, Eq. (**??**),

$$\langle t^2 \rangle \langle \Omega^2 \rangle = \frac{M^4}{4} \kappa_c \geq \frac{1}{4}. \tag{1.65}$$

We have introduced a factor of chirp κ_c, equal to

$$\kappa_c = 1 + \frac{M^4}{4\langle t^2 \rangle_0^2} \left[\left. \frac{d^2\phi}{d\Omega^2} \right|_0 \right]^2 \tag{1.66}$$

in case of a frequency chirp and constant spectrum, or

$$\kappa_c = 1 + \frac{M^4}{4\langle \Omega^2 \rangle_0^2} \left[\left. \frac{d^2\varphi}{dt^2} \right|_0 \right]^2 \tag{1.67}$$

in case of a temporal chirp and constant pulse envelope.

In summary, using the *mean square deviation* to define the pulse duration and bandwidth:

- the duration—bandwidth product $\sqrt{\langle t^2 \rangle \langle \Omega^2 \rangle}$ is minimum (0.5) for a Gaussian pulse shape, without phase modulation.
- For any pulse shape, one can define a shape factor M^2 equal to the minimum duration—bandwidth product for that particular shape.
- Any quadratic phase modulation — or linear chirp — whether in frequency or time, increases the bandwidth duration product by a chirp factor κ_c. The latter increases proportionally to the second derivative of the phase modulation, whether in time or in frequency.

1.2 Pulse propagation

So far we have considered only temporal and spectral characteristics of light pulses. In this subsection we shall be interested in the propagation of such pulses through matter. This is the situation one always encounters when working with electromagnetic wave packets (at least until somebody succeeds in building a suitable trap). The electric field, now considered in its temporal and spatial dependence, is again a suitable quantity for the description of the propagating wave packet. In view of the optical materials that will be investigated, we can neglect external charges and currents and confine ourselves to nonmagnetic permeabilities and uniform media. A wave equation can be derived for the electric field vector **E** from Maxwell equations (see for instance Ref. [12]) which in Cartesian coordinates reads

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(x,y,z,t)=\mu_0\frac{\partial^2}{\partial t^2}\mathbf{P}(x,y,z,t)}, \tag{1.68}$$

where μ_0 is the magnetic permeability of free space. The source term of Eq. (1.68) contains the polarization **P** and describes the influence of the medium on the field as well as the response of the medium. Usually the polarization is decomposed into two parts:

$$\mathbf{P}=\mathbf{P}^L+\mathbf{P}^{NL}. \tag{1.69}$$

The decomposition of Eq. (1.69) is intended to distinguish a polarization that varies linearly ($\mathbf{P}^L$) from one that varies nonlinearly ($\mathbf{P}^{NL}$) with the field. Historically, $\mathbf{P}^L$ represents the medium response in the frame of "ordinary" optics, e.g., classical optics [13], and is responsible for effects such as diffraction, dispersion, refraction, linear losses and linear gain. Frequently, these processes can be attributed to the action of a host material which in turn may contain sources of a nonlinear polarization $\mathbf{P}^{NL}$. The latter is responsible for nonlinear optics [14, 15, 16] which includes, for instance, saturable absorption and gain, harmonic generation and Raman processes.

As will be seen in Chapters **??** and **??**, both $\mathbf{P}^L$ and in particular $\mathbf{P}^{NL}$ are often related to the electric field by complicated differential equations. One reason is that no physical phenomenon can be truly instantaneous. In this chapter we will omit $\mathbf{P}^{NL}$. Depending on the actual problem under consideration, $\mathbf{P}^{NL}$ will have to be specified and added to the wave equation as a source term.

1.2.1 The reduced wave equation

Equation (1.68) is of rather complicated structure and in general can solely be solved by numerical methods. However, by means of suitable approximations and simplifications, one can derive a "reduced wave equation" which will enable us to deal with many practical pulse propagation problems in a rather simple way. We assume the electric field to be linearly polarized and propagating in the z-direction as a plane wave, i.e., the field is uniform in the transverse x, y direction. The wave equation has now been simplified to:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E(z,t) = \mu_0 \frac{\partial^2}{\partial t^2} P^L(z,t) \tag{1.70}$$

As known from classical electrodynamics [12] the linear polarization of a medium is related to the field through the dielectric susceptibility χ. In the frequency domain we have

$$\tilde{P}^L(\Omega, z) = \epsilon_0 \; \chi(\Omega)\tilde{E}(\Omega, z) \tag{1.71}$$

which is equivalent to a convolution integral in the time domain

$$P^L(t,z) = \epsilon_0 \int_{-\infty}^{t} dt' \, \chi(t') E(z, t-t'). \tag{1.72}$$

Here ϵ_0 is the permittivity of free space. The finite upper integration limit, t, expresses the fact that the response of the medium must be causal. For a nondispersive medium (which implies an "infinite bandwidth" for the susceptibility, $\chi(\Omega)$ =const.) the medium response is instantaneous, i.e., memory free. In general, $\chi(t)$ describes a finite response time of the medium which, in the frequency domain, means nonzero dispersion. This simple fact has important implications for the propagation of short pulses and time varying radiation in general. We will refer to this point several times in later chapters — in particular when dealing with coherent interaction.

The Fourier transform of (1.70) together with (1.71) yields

$$\boxed{\left[\frac{\partial^2}{\partial z^2} + \Omega^2 \epsilon(\Omega)\mu_0\right] \tilde{E}(z,\Omega) = 0} \tag{1.73}$$

where we have introduced the dielectric constant

$$\epsilon(\Omega) = [1 + \chi(\Omega)]\epsilon_0. \tag{1.74}$$

For now we will assume a real susceptibility and dielectric constant. Later we will discuss effects associated with complex quantities. The general solution of (1.73) for the propagation in the $+z$ direction is

$$\tilde{E}(\Omega, z) = \tilde{E}(\Omega, 0)e^{-ik(\Omega)z}, \tag{1.75}$$

where the propagation constant $k(\Omega)$ is determined by the dispersion relation of linear optics

$$k^2(\Omega) = \Omega^2 \epsilon(\Omega)\mu_0 = \frac{\Omega^2}{c^2} n^2(\Omega), \tag{1.76}$$

and $n(\Omega)$ is the refractive index of the material. For further consideration we expand $k(\Omega)$ about the carrier frequency ω_ℓ

$$k(\Omega) = k(\omega_\ell) + \delta k, \tag{1.77}$$

where

$$\delta k = \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} (\Omega - \omega_\ell) + \frac{1}{2} \left.\frac{d^2 k}{d\Omega^2}\right|_{\omega_\ell} (\Omega - \omega_\ell)^2 + \ldots \tag{1.78}$$

and write Eq. (1.75) as

$$\tilde{E}(\Omega, z) = \tilde{E}(\Omega, 0)e^{-ik_\ell z}e^{-i\,\delta k\, z}, \tag{1.79}$$

where $k_\ell^2 = \omega_\ell^2 \epsilon(\omega_\ell)\mu_0 = \omega_\ell^2 n^2(\omega_\ell)/c^2$. In most practical cases of interest, the Fourier amplitude will be centered around a mean wave vector k_ℓ, and will have appreciable values only in an interval Δk small compared to k_ℓ. In analogy to the introduction of an envelope function slowly varying in time, after the separation of a rapidly oscillating term, cf. Eqs. (1.11)– (1.14), we can define now an amplitude which is slowly varying in the spatial coordinate

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{E}(\Omega + \omega_\ell, 0)e^{-i\,\delta k\,z}. \tag{1.80}$$

Again, for this concept to be useful we must require that

$$\left|\frac{d}{dz}\tilde{\mathcal{E}}(\Omega, z)\right| \ll k_\ell \left|\tilde{\mathcal{E}}(\Omega, z)\right| \tag{1.81}$$

which implies a sufficiently small wave number spectrum

$$\left|\frac{\Delta k}{k_\ell}\right| \ll 1. \tag{1.82}$$

In other words, the pulse envelope must not change significantly while travelling through a distance comparable with the wavelength $\lambda_\ell = 2\pi/\omega_\ell$. Fourier transforming of Eq. (1.79)) into the time domain gives

$$\tilde{E}(t, z) = \frac{1}{2}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} d\Omega\,\tilde{E}(\Omega, 0)e^{-i\,\delta k\,z}e^{i(\Omega-\omega_\ell)t}\right\} e^{i(\omega_\ell t - k_\ell z)} \tag{1.83}$$

which can be written as

$$\boxed{\tilde{E}(t, z) = \frac{1}{2}\tilde{\mathcal{E}}(t, z)e^{i(\omega_\ell t - k_\ell z)}} \tag{1.84}$$

where $\tilde{\mathcal{E}}(t, z)$ is now the envelope varying slowly in space and time, defined by the term in the curled brackets in Eq. (1.83).

Further simplification of the wave equation requires a corresponding equation for $\tilde{\mathcal{E}}$ utilizing the envelope properties. Only a few terms in the expansion of $k(\Omega)$ and $\epsilon(\Omega)$, respectively, will be considered. To this effect we expand $\epsilon(\Omega)$ as series around ω_ℓ, leading to the following form for the linear polarization (1.71)

$$\tilde{P}^L(\Omega, z) = \left(\epsilon(\omega_\ell) - \epsilon_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^n\epsilon}{d\Omega^n}\bigg|_{\omega_\ell}(\Omega - \omega_\ell)^n\right)\tilde{E}(\Omega, z). \tag{1.85}$$

In terms of the pulse envelope, the above expression corresponds in the time domain to

$$\begin{aligned}\tilde{P}^L(t, z) &= \frac{1}{2}\bigg\{[\epsilon(\omega_\ell) - \epsilon_0]\tilde{\mathcal{E}}(t, z) \\ &+ \sum_{n=1}^{\infty}(-i)^n\frac{\epsilon^{(n)}(\omega_\ell)}{n!}\frac{\partial^n}{\partial t^n}\tilde{\mathcal{E}}(t, z)\bigg\}e^{i(\omega_\ell t - k_\ell z)},\end{aligned} \tag{1.86}$$

where $\epsilon^{(n)}(\omega_\ell) = \frac{\partial^n}{\partial\Omega^n}\epsilon\Big|_{\omega_\ell}$. The term in the curled brackets defines the slowly varying envelope of the polarization, $\tilde{\mathcal{P}}^L$. The next step is to replace the electric field and the polarization in the wave equation (1.70) by Eq. (1.83) and Eq. (1.86), respectively. We transfer thereafter to a coordinate system (η, ξ) moving with the group velocity $v_g = \left(\frac{dk}{d\Omega}\Big|_{\omega_\ell}\right)^{-1}$, which is the standard transformation to a "retarded" frame of reference:

$$\xi = z \qquad \eta = t - \frac{z}{v_g} \tag{1.87}$$

and

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi} - \frac{1}{v_g}\frac{\partial}{\partial \eta}; \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}. \tag{1.88}$$

A straightforward calculation leads to the final result:

$$\frac{\partial}{\partial \xi}\tilde{\mathcal{E}} - \frac{i}{2}k_\ell''\frac{\partial^2}{\partial \eta^2}\tilde{\mathcal{E}} + \mathcal{D} = -\frac{i}{2k_\ell}\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi} - \frac{2}{v_g}\frac{\partial}{\partial \eta}\right)\tilde{\mathcal{E}} \tag{1.89}$$

The quantity

$$\begin{aligned}\mathcal{D} &= -\frac{i\mu_0}{2k_\ell}\sum_{n=3}^{\infty}\frac{(-i)^n}{n!}\Big[\omega_\ell^2\epsilon^{(n)}(\omega_\ell) - 2n\omega_\ell\epsilon^{(n-1)}(\omega_\ell) \\ &+ \; n(n-1)\epsilon^{(n-2)}(\omega_\ell)\Big]\frac{\partial^n}{\partial \eta^n}\tilde{\mathcal{E}}\end{aligned} \tag{1.90}$$

contains dispersion terms of higher order, and has been derived by taking directly the second order derivative of the polarization defined by the product of envelope and fast oscillating terms in Eq. (1.86). The indices of the three resulting terms have been re-defined to factor out a single derivative of order (n) of the field envelope. The second derivative of k:

$$\begin{aligned}k_\ell'' &= \frac{\partial^2 k}{\partial \Omega^2}\Big|_{\omega_\ell} = -\frac{1}{v_g^2}\frac{dv_g}{d\Omega}\Big|_{\omega_\ell} \\ &= \frac{1}{2k_\ell}\left[\frac{2}{v_g^2} - 2\mu_0\epsilon(\omega_\ell) - 4\omega_\ell\mu_0\epsilon^{(1)}(\omega_\ell) - \omega_\ell^2\mu_0\epsilon^{(2)}(\omega_\ell)\right]\end{aligned} \tag{1.91}$$

is the group velocity dispersion (GVD) parameter. It should be mentioned that the GVD is usually defined as the derivative of v_g with respect to λ, $dv_g/d\lambda$, related to k'' through

$$\frac{dv_g}{d\lambda} = \frac{\Omega^2 v_g^2}{2\pi c}\frac{d^2k}{d\Omega^2}. \tag{1.92}$$

So far we have not made any approximations and the structure of Eq. (1.89) is still rather complex. However, we can exploit at this point the envelope properties (1.14) and (1.81), which, in this particular situation, imply:

$$\left|\frac{1}{k_\ell}\left(\frac{\partial}{\partial\xi}-\frac{2}{v_g}\frac{\partial}{\partial\eta}\right)\tilde{\mathcal{E}}\right| = \left|\frac{1}{k_\ell}\left(\frac{\partial}{\partial z}-\frac{1}{v_g}\frac{\partial}{\partial t}\right)\tilde{\mathcal{E}}\right| \ll \left|\tilde{\mathcal{E}}\right| \tag{1.93}$$

The right–hand side of (1.89) can thus be neglected if the prerequisites for introducing pulse envelopes are fulfilled. This procedure is called slowly varying envelope approximation (SVEA) and reduces the wave equation to first–order derivatives with respect to the spatial coordinate.

If the propagation of very short pulses is computed over long distances, the cumulative error introduced by neglecting the right hand side of Eq. (1.89) may be significant. In those cases, a direct numerical treatment of the second order wave equation is required.

Further simplifications are possible for a very broad class of problems of practical interest, where the dielectric constant changes slowly over frequencies within the pulse spectrum. In those cases, terms with $n \geq 3$ can be omitted too ($\mathcal{D} = 0$), leading to a greatly simplified reduced wave equation:

$$\boxed{\frac{\partial}{\partial\xi}\tilde{\mathcal{E}}(\eta,\xi) - \frac{i}{2}k''_\ell\frac{\partial^2}{\partial\eta^2}\tilde{\mathcal{E}}(\eta,\xi) = 0} \tag{1.94}$$

which describes the evolution of the complex pulse envelope as it propagates through a loss-free medium with GVD. The reader will recognize the structure of the one–dimensional Schrödinger equation.

1.2.2 Retarded frame of reference

In the case of zero GVD [$k''_\ell = 0$ in Eq. (1.94)], the pulse envelope does not change at all in the system of local coordinates (η, ξ). This illustrates the usefulness of introducing a coordinate system moving at the group velocity. In the laboratory frame, the pulse travels at the group velocity without any distortion.

In dealing with short pulses as well as in dealing with white light (see Chapter **??**) the appropriate "retarded frame of reference" is moving at the *group* rather than at the *wave* (*phase*) velocity. Indeed, while a monochromatic wave of frequency Ω travels at the phase velocity $v_p(\Omega) = c/n(\Omega)$, it is the superposition of many such waves with differing phase velocities that leads to a wave packet (pulse) propagating with the group velocity. The importance of the frame of reference moving at the group velocity is such that, in the following chapters, the notation z and t will be substituted for ξ and η, unless the laboratory frame is explicitly specified.

Some propagation problems — such as the propagation of coupled waves in nonlinear crystals discussed in Chapter **??** — are more appropriately treated in the frequency domain. As a simple exercise, let us derive the group velocity directly from the solution of the wave equation in the form of Eq. (1.79)

$$\tilde{E}(\Omega, z) = \tilde{E}(\Omega, 0)e^{-ik_\ell z}e^{-i\,\delta k\, z}. \tag{1.95}$$

The Fourier transform amplitude amplitude $E(\Omega, 0)$ represented on the top left of Fig. 1.7 is not changed by propagation. On the top right, the time domain representation of the pulse, or the inverse transform of $E(\Omega, 0)$, is centered at $t = 0$ (solid line). We assume that the expansion of the wave

Figure 1.7: The Fourier transform amplitude ($E(\Omega, 0)$ is sketched in the upper left, and the corresponding field in the time domain on the upper right (solid line). The lower part of the figure displays the field amplitudes, $\mathcal{E}(\Omega)$ on the left, centered at the origin of the frequency scale, and the corresponding inverse Fourier transform $\mathcal{E}(t)$. Propagation in the frequency domain is obtained by multiplying the field at $z = 0$ by the phase factor $\exp(-i\tau_d\Omega)$, where $\tau_d = z/v_g$ is the group delay. In the time domain, this corresponds to delaying the pulse by an amount τ_d (right). The delayed fields $|E(z, t)|$ and $\mathcal{E}(z, t)$ are shown in dotted lines on the right of the figure.

vector $k(\Omega)$, Eq. (1.77), can be terminated after the linear term, that is

$$\delta k = \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} (\Omega - \omega_\ell) \tag{1.96}$$

The inverse Fourier-transform of Eq. (1.95) now yields

$$\begin{aligned} \tilde{E}(t,z) &= e^{-ik_\ell z} \int_{-\infty}^{\infty} \tilde{E}(\Omega,0) \exp\left[-i \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} (\Omega - \omega_\ell) z\right] e^{i\Omega t} d\Omega \qquad (1.97) \\ &= e^{i(\omega_\ell t - k_\ell z)} \int_{-\infty}^{\infty} \tilde{E}(\Omega' + \omega_\ell, 0) \exp\left[i\left(t - \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} z\right)\Omega'\right] d\Omega' \end{aligned}$$

where we substituted $\Omega = \Omega' + \omega_\ell$ to obtain the last equation. This equation is just the inverse Fourier-transform of the field spectrum shifted to the origin (i.e., the spectrum of the envelope $\tilde{\mathcal{E}}(\Omega)$, represented on the lower left of Fig. 1.7) with the Fourier variable "time" now given by $t - \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} z$. Carrying out the transform yields

$$\tilde{E}(t,z) = \frac{1}{2}\tilde{\mathcal{E}}(t,z)e^{i(\omega_\ell t - k_\ell z)} = \frac{1}{2}\tilde{\mathcal{E}}\left(t - \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} z, 0\right) e^{i(\omega_\ell t - k_\ell z)}. \tag{1.98}$$

We have thus the important result that, in the time domain, the light pulse has been delayed by an amount ($\tau_d = \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} z$) proportional to distance. Within the approximation that the wave vector is a linear function of frequency, the pulse is seen to propagate without distortion with a constant group velocity v_g given by either of the three expressions:

$$\boxed{\begin{aligned} \frac{1}{v_g} &= \left.\frac{dk}{d\Omega}\right|_{\omega_\ell} && (1.99) \\ \frac{1}{v_g} &= \frac{n_0}{c} + \frac{\omega_\ell}{c}\left.\frac{dn}{d\Omega}\right|_{\omega_\ell} && (1.100) \\ \frac{1}{v_g} &= \frac{n_0}{c} - \frac{\lambda}{c}\left.\frac{dn}{d\lambda}\right|_{\lambda}. && (1.101) \end{aligned}}$$

The first term in Eqs. (1.100) and (1.101) represent the phase delay per unit length, while the second term in these equations is the change in carrier to envelope phase per unit length. We note that the dispersion of the wave vector ($dk/d\Omega$) or of the index of refraction ($dn/d\lambda$) is responsible for a difference between the phase velocity $v_p = c/n_0$ and the group velocity v_g. In a frame of reference moving at the velocity v_g, $\tilde{\mathcal{E}}(z,t)$ remains identically unchanged. Pulse distortions thus only result from high order (higher than 1) terms in the Taylor series expansion of $k(\Omega)$. For this reason, most pulse propagation problems are treated in a retarded frame of reference, moving at the velocity v_g.

Forward/Backward propagating waves

We consider an ultrashort pulse plane wave propagating through a dielectric medium. Before the arrival of the pulse, there are no induced dipoles, and for the index of refraction we assume that of a vacuum ($n = 1$). As the dipoles are driven into motion by the first few cycles of the pulse, the index of refraction changes to the value n of the dielectric. One consequence of this causal phenomenon is the "precursor" predicted by Sommerfeld and Brillouin, see for example [12]. One might wonder if the discontinuity in index created by a short and intense pulse should not lead to a reflection for a portion of the pulse? This is an important question regarding the validity of the first order approximation to Maxwell's propagation equations. If, at $t = 0$, a short wave packet is launched in the $+z$ direction in a homogeneous medium, is it legitimate to assume that there will be no pulse generated in the opposite direction?

The answer that we give in this section is that, in the framework of Maxwell's second order equation and a linear polarization, there is no such "induced reflection". This property extends even to the nonlinear polarization created by the interaction of the light with a two-level system.

If we include the non-resonant part of the linear polarization in the index of refraction n (imaginary part of n), the remainder polarization P including all nonlinear and resonant interaction effects, adding a phenomenological scattering term σ leads to the following form of the second order wave equation:

$$\left(\frac{\partial^2}{\partial z^2}\tilde{E} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{E} = \mu_0\frac{\partial^2}{\partial t^2}\tilde{P} + \frac{n\sigma}{c}\frac{\partial}{\partial t}\tilde{E} \tag{1.102}$$

The polarization appearing in the right hand side can be instantaneous, or be the solution of a differential equation as in the case of most interactions with resonant atomic or molecular systems. Resonant light-matter interactions will be studied in detail in Chapters **??** and **??**. The wave equation Eq. (1.102) can be written as a product of a forward and backward propagating operator. Instead of the variables t and z, it is more convenient to use the retarded time variable corresponding to the two possible wave velocities

$\pm c/n$:

$$\begin{aligned} s &= t - \frac{n}{c} z \\ r &= t + \frac{n}{c} z. \end{aligned} \tag{1.103}$$

In the new variables, Maxwell's equation (1.102) becomes:

$$\frac{\partial^2}{\partial s \partial r} \tilde{E} = \frac{c^2}{n^2} \left\{ \frac{\mu_0}{4} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right)^2 \tilde{P} + \frac{n\sigma}{c} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right) \right\} \tilde{E}. \tag{1.104}$$

We seek a solution in the form of a forward and a backward propagating field of amplitude $\tilde{\mathcal{E}}_F$ and $\tilde{\mathcal{E}}_B$:

$$\tilde{E} = \frac{1}{2} \tilde{\mathcal{E}}_F e^{i\omega_\ell s} + \frac{1}{2} \tilde{\mathcal{E}}_B e^{i\omega_\ell r}. \tag{1.105}$$

Substitution into Maxwell's Eq. (1.102):

$$\begin{aligned} & e^{i\omega_\ell s} \left[2i\omega_\ell \frac{\partial}{\partial r} + \frac{\partial^2}{\partial s \partial r} + \frac{c\sigma}{2n} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} + 2i\omega_\ell \right) \right] \frac{1}{2} \tilde{\mathcal{E}}_F \\ + \quad & e^{i\omega_\ell r} \left[2i\omega_\ell \frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial r} + \frac{c\sigma}{2n} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} + 2i\omega_\ell \right) \right] \frac{1}{2} \tilde{\mathcal{E}}_B \\ = \quad & -\frac{\mu_0 c^2}{4n^2} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right)^2 \tilde{P}, \end{aligned} \tag{1.106}$$

which we re-write in an abbreviated way using the differential operators $\mathcal{L}$ and $\mathcal{M}$ for the forward and backward propagating waves, respectively:

$$\mathcal{L} \tilde{\mathcal{E}}_F e^{i\omega_\ell s} + \mathcal{M} \tilde{\mathcal{E}}_B e^{i\omega_\ell r} = -\frac{\mu_0 c^2}{4n^2} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right)^2 \tilde{P}. \tag{1.107}$$

In the case of a linear medium, the forward and backward wave travel independently. If, as initial condition, we choose $\tilde{\mathcal{E}}_B = 0$ along the line $r + s = 0$ ($t = 0$), there will be no back scattered wave. If the polarization is written as a slowly varying amplitude:

$$\tilde{P} = \frac{1}{2} \tilde{\mathcal{P}}_F e^{i\omega_\ell s} + \frac{1}{2} \tilde{\mathcal{P}}_B e^{i\omega_\ell r}, \tag{1.108}$$

the equations for the forward and backward propagating wave also separate if $\tilde{\mathcal{P}}_F$ is only a function of $\tilde{\mathcal{E}}_F$, and $\tilde{\mathcal{P}}_B$ only a function of $\tilde{\mathcal{E}}_B$. This is because a source term for $\tilde{\mathcal{P}}_B$ can only be formed by a "grating" term, which involves a product of $\tilde{\mathcal{E}}_B\tilde{\mathcal{E}}_F$. It applies to a polarization created by near resonant interaction with a two-level system, using the semi-classical approximation, as will be considered in Chapters **??** and **??**. The separation between forward and backward travelling waves has been demonstrated by Eilbeck [17, 18] outside of the slowly-varying approximation. Within the slowly varying approximation, we generally write that the second derivative with respect to time of the polarization as $-\omega_\ell^2\tilde{\mathcal{P}}$, and therefore, the forward and backward propagating waves are still uncoupled, even when $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(\tilde{\mathcal{E}}_F, \tilde{\mathcal{E}}_B)$, provided there is only a forward propagating beam as initial condition.

1.2.3 Dispersion

For nonzero GVD ($k_\ell'' \neq 0$) the propagation problem (1.94) can be solved either directly in the time or in the frequency domain. In the first case, the solution is given by a Poisson-integral [19] which here reads

$$\tilde{\mathcal{E}}(t,z) = \frac{1}{\sqrt{2\pi i k_\ell'' z}} \int_{-\infty}^{t} \tilde{\mathcal{E}}(t', z=0) \exp\left(i\frac{(t-t')^2}{2k_\ell'' z}\right) dt' \tag{1.109}$$

As we will see in subsequent chapters, it is generally more convenient to treat linear pulse propagation through transparent linear media in the frequency domain, since only the phase factor of the envelope $\tilde{\mathcal{E}}(\Omega)$ is affected by propagation.

It follows directly from the solution of Maxwell's equations in the frequency domain [for instance Eqs. (1.75) and (1.80)] that the spectral envelope after propagation through a thickness z of a linear transparent material is given by:

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{\mathcal{E}}(\Omega, 0) \exp\left(-\frac{i}{2}k_\ell''\Omega^2 z - \frac{i}{3!}k_\ell'''\Omega^3 z - \ldots\right). \tag{1.110}$$

Thus we have for the temporal envelope

$$\tilde{\mathcal{E}}(t, z) = \mathcal{F}^{-1}\left\{\tilde{\mathcal{E}}(\Omega, 0) \exp\left(-\frac{i}{2}k_\ell''\Omega^2 z - \frac{i}{3!}k_\ell'''\Omega^3 z - \ldots\right)\right\}. \tag{1.111}$$

If we limit the Taylor expansion of k to the GVD term k_ℓ'', we find that an initially bandwidth-limited pulse develops a spectral phase with a quadratic frequency dependence, resulting in chirp.

We had defined a "chirp coefficient"

$$\kappa_c = 1 + \frac{M^4}{4\langle t^2\rangle_0^2}\left[\left.\frac{d\phi}{d\Omega}\right|_{\omega_\ell}\right]^2$$

when considering in Section 1.1.4 the influence of quadratic chirp on the uncertainty relation Eq. (1.65) based on the successive moments of the field distribution. In the present case, we can identify the phase modulation:

$$\left.\frac{d\phi}{d\Omega}\right|_{\omega_\ell} = -k''_\ell z \tag{1.112}$$

Since the spectrum (in amplitude) of the pulse $|\tilde{\mathcal{E}}(\Omega, z)|^2$ remains constant [as shown for instance in Eq. (1.110)], the spectral components responsible for chirp must appear at the expense of the envelope shape, which has to become broader.

At this point we want to introduce some useful relations for the characterization of the dispersion. The dependence of a dispersive parameter can be given as a function of either the frequency Ω or the vacuum wavelength λ. The first, second and third order derivatives are related to each other by

$$\frac{d}{d\Omega} = -\frac{\lambda^2}{2\pi c}\frac{d}{d\lambda} \tag{1.113}$$

$$\frac{d^2}{d\Omega^2} = \frac{\lambda^2}{(2\pi c)^2}\left(\lambda^2\frac{d^2}{d\lambda^2} + 2\lambda\frac{d}{d\lambda}\right) \tag{1.114}$$

$$\frac{d^3}{d\Omega^3} = -\frac{\lambda^3}{(2\pi c)^3}\left(\lambda^3\frac{d^3}{d\lambda^3} + 6\lambda^2\frac{d^2}{d\lambda^2} + 6\lambda\frac{d}{d\lambda}\right) \tag{1.115}$$

The dispersion of the material is described by either the frequency dependence $n(\Omega)$ or the wavelength dependence $n(\lambda)$ of the index of refraction. The derivatives of the propagation constant used most often in pulse propagation problems, expressed in terms of the index n, are:

$$\frac{dk}{d\Omega} = \frac{n}{c} + \frac{\Omega}{c}\frac{dn}{d\Omega} = \frac{1}{c}\left(n - \lambda\frac{dn}{d\lambda}\right) \tag{1.116}$$

$$\frac{d^2k}{d\Omega^2} = \frac{2}{c}\frac{dn}{d\Omega} + \frac{\Omega}{c}\frac{d^2n}{d\Omega^2} = \left(\frac{\lambda}{2\pi c}\right)\frac{1}{c}\left(\lambda^2\frac{d^2n}{d\lambda^2}\right) \tag{1.117}$$

$$\frac{d^3k}{d\Omega^3} = \frac{3}{c}\frac{d^2n}{d\Omega^2} + \frac{\Omega}{c}\frac{d^3n}{d\Omega^3} = -\left(\frac{\lambda}{2\pi c}\right)^2\frac{1}{c}\left(3\lambda^2\frac{d^2n}{d\lambda^2} + \lambda^3\frac{d^3n}{d\lambda^3}\right) \tag{1.118}$$

The second equation, Eq. (1.117), defining the group velocity dispersion (GVD) is the frequency derivative of $1/v_g$. Multiplied by the propagation length L, it describes the frequency dependence of the group delay. It is sometimes expressed in $\mathrm{fs}^2\ \mu\mathrm{m}^{-1}$.

A positive GVD corresponds to

$$\frac{d^2k}{d\Omega^2} > 0 \tag{1.119}$$

1.2.4 Gaussian pulse propagation

For a more quantitative picture of the influence that GVD has on the pulse propagation we consider the linearly chirped Gaussian pulse of Eq. (1.33)

$$\tilde{\mathcal{E}}(t, z = 0) = \mathcal{E}_0 e^{-(1+ia)(t/\tau_{G0})^2} = \mathcal{E}_0 e^{-(t/\tau_{G0})^2} e^{i\varphi(t,z=0)}$$

entering the sample. To find the pulse at an arbitrary position z, we multiply the field spectrum, Eq. (1.35), with the propagator $\exp\left(-i\frac{1}{2}k_\ell''\Omega^2 z\right)$ as done in Eq. (1.110), to obtain

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{A}_0 e^{-x\Omega^2} e^{iy\Omega^2} \tag{1.120}$$

where

$$x = \frac{\tau_{G0}^2}{4(1+a^2)} \tag{1.121}$$

and

$$y(z) = \frac{a\tau_{G0}^2}{4(1+a^2)} - \frac{k_\ell'' z}{2}. \tag{1.122}$$

$\tilde{A}_0$ is a complex amplitude factor which we will not consider in what follows and τ_{G0} describes the pulse duration at the sample input. The time dependent electric field that we obtain by Fourier transforming Eq. (1.120) can be written as

$$\tilde{\mathcal{E}}(t, z) = \tilde{A}_1 \exp\left\{-\left(1 + i\frac{y(z)}{x}\right)\left(\frac{t}{\sqrt{\frac{4}{x}[x^2 + y^2(z)]}}\right)^2\right\}. \tag{1.123}$$

Obviously, this describes again a linearly chirped Gaussian pulse. For the "pulse duration" (note $\tau_p = \sqrt{2\ln 2}\,\tau_G$) and phase at position z we find

$$\tau_G(z) = \sqrt{\frac{4}{x}[x^2 + y^2(z)]} \tag{1.124}$$

and

$$\varphi(t, z) = -\frac{y(z)}{4[x^2 + y^2(z)]} t^2. \tag{1.125}$$

Let us consider first an initially unchirped input pulse ($a = 0$). The pulse duration and chirp parameter develop as:

$$\tau_G(z) = \tau_{G0}\sqrt{1+\left(\frac{z}{L_d}\right)^2} \quad (1.126)$$

$$\frac{\partial^2}{\partial t^2}\varphi(t,z) = \left(\frac{1}{\tau_{G0}^2}\right)\frac{2z/L_d}{1+(z/L_d)^2}. \quad (1.127)$$

We have defined a characteristic length:

$$L_d = \frac{\tau_{G0}^2}{2k_\ell''}. \quad (1.128)$$

For later reference let also us introduce a so-called dispersive length defined as

$$L_D = \frac{\tau_{p0}^2}{k_\ell''} \quad (1.129)$$

where for Gaussian pulses $L_D \approx 2.77 L_d$. Bandwidth limited Gaussian pulses double their length after propagation of about $0.6 L_D$. For propagation lengths $z \gg L_d$ the pulse broadening of an unchirped input pulse as described by Eq. (1.126) can be simplified to

$$\frac{\tau_G(z)}{\tau_{G0}} \approx \frac{z}{|L_d|} = \frac{2|k_\ell''|}{\tau_{G0}^2}z. \quad (1.130)$$

It is interesting to compare the result of Eq. (1.126) with that of Eq. (1.63), where we used the second moment as a measure for the pulse duration. Since the Gaussian is the shape for minimum uncertainty [Eq. (1.57)], and since $d^2\phi/d\Omega^2 = -k''z$, one can derive the evolution equation for the mean square deviation of a Gaussian pulse in a dielectric medium:

$$\langle t^2\rangle = \langle t^2\rangle_0 + \left.\frac{d^2\phi}{d\Omega^2}\right|_0 \langle\Omega^2\rangle_0 = \langle t^2\rangle = \langle t^2\rangle_0 + \frac{(k'')^2 z^2}{\langle t^2\rangle_0}. \quad (1.131)$$

The latter equations reduces to Eq. (1.126) by substituting the relations between mean square deviations and Gaussian widths [Eq. (1.58)]. If the input pulse is chirped ($a \neq 0$) two different behaviors can occur depending on the relative sign of a and k_ℓ''. In the case of opposite sign, $y^2(z)$ increases monotonously resulting in pulse broadening, cf. Eq. (1.124). If a and k_ℓ''

have equal sign $y^2(z)$ decreases until it becomes zero after a propagation distance

$$z_c = \frac{\tau_{G0}^2 a}{2|k_\ell''|(1+a^2)}. \tag{1.132}$$

At this position the pulse reaches its shortest duration

$$\tau_G(z_c) = \tau_{Gmin} = \frac{\tau_{G0}}{\sqrt{1+a^2}} \tag{1.133}$$

and the time dependent phase according to Eq. (1.125) vanishes. From here on the propagation behavior is that of an unchirped input pulse of duration τ_{Gmin}, that is, the pulse broadens and develops a time-dependent phase. The larger the input chirp ($|a|$), the shorter the minimum pulse duration that can be obtained [see Eq. (1.133)]. The underlying reason is that the excess bandwidth of a chirped pulse is converted into a narrowing of the envelope by chirp compensation, until the Fourier limit is reached. The whole procedure including the impression of chirp on a pulse will be treated in Chapter **??** in more detail.

There is a complete analogy between the propagation (diffraction) effects of a spatially Gaussian beam and the temporal evolution of a Gaussian pulse in a dispersive medium. For instance, the pulse duration and the slope of the chirp follow the same evolution with distance as the waist and curvature of a Gaussian beam, as detailed at the end of this chapter. A linearly chirped Gaussian pulse in a dispersive medium is completely characterized by the position and (minimum) duration of the unchirped pulse, just as a spatially Gaussian beam is uniquely defined by the position and size of its waist. To illustrate this point, let us consider a linearly chirped pulse whose “duration” τ_G and chirp parameter a are known at a certain position z_1. The position z_c of the minimum duration (unchirped pulse) is found again by setting $y = 0$ in Eq. (1.122):

$$z_c = z_1 + \frac{\tau_G^2}{2k_\ell''}\frac{a}{1+a^2} = z_1 + a\frac{\tau_{Gmin}^2}{2k_\ell''}. \tag{1.134}$$

The position z_c is after z_1 if a and k_ℓ'' have the same sign[2]; before z_1 if they have opposite sign. All the temporal characteristics of the pulse are most conveniently defined in terms of the distance $L = z - z_c$ to the point of zero chirp, and the minimum duration τ_{Gmin}. This is similar to Gaussian beam propagation where the location of the beam waist often serves as reference.

[2]For instance, an initially downchirped ($a > 0$) pulse at $z = z_c$ will be compressed in a medium with positive dispersion ($k'' > 0$).

The chirp parameter a and the pulse "duration" τ_G at any point L are then simply given by

$$a(L) = L/L_d \tag{1.135}$$

$$\tau_G(L) = \tau_{Gmin}\sqrt{1+[a(L)]^2} \tag{1.136}$$

where the dispersion parameter $L_d = \tau_{Gmin}^2/(2|k_\ell''|)$. The pulse duration bandwidth product varies with distance L as

$$c_B(L) = \frac{2\ln 2}{\pi}\sqrt{1+[a(L)]^2} \tag{1.137}$$

To summarize, Fig. (1.8) illustrates the behavior of a linearly chirped Gaussian pulse as it propagates through a dispersive sample.

Simple physical consideration can lead directly to a crude approximation for the maximum broadening that a bandwidth limited pulse of duration τ_p and spectral width $\Delta\omega_p$ will experience. Each group of waves centered around a frequency Ω travels with its own group velocity $v_g(\Omega)$. The difference of group velocities over the pulse spectrum becomes then:

$$\Delta v_g = \left[\frac{dv_g}{d\Omega}\right]_{\omega_\ell}\Delta\omega_p. \tag{1.138}$$

Accordingly, after a travel distance L the pulse spread can be as large as

$$\Delta\tau_p = \left|\Delta\left(\frac{L}{v_g}\right)\right| \approx \frac{L}{V_g^2}|\Delta v_g| \tag{1.139}$$

which, by means of Eqs. (1.91) and (1.138), yields:

$$\Delta\tau_p = L|k_\ell''|\Delta\omega_p. \tag{1.140}$$

Approximating $\tau_p \approx \Delta\omega_p^{-1}$, a characteristic length after which a pulse has approximately doubled its duration can now be estimated as:

$$L_D' = \frac{1}{|k_\ell''|\Delta\omega_p^2}. \tag{1.141}$$

Measuring the length in meter and the spectral width in nm the GVD of materials is sometimes given in fs/(m nm) which pictorially describes the pulse broadening per unit travel distance and unit spectral width. From Eq. (1.140) we find for the corresponding quantity

$$\boxed{\frac{\Delta\tau_p}{L\Delta\lambda} = 2\pi\frac{c}{\lambda_\ell^2}|k_\ell''|.} \tag{1.142}$$

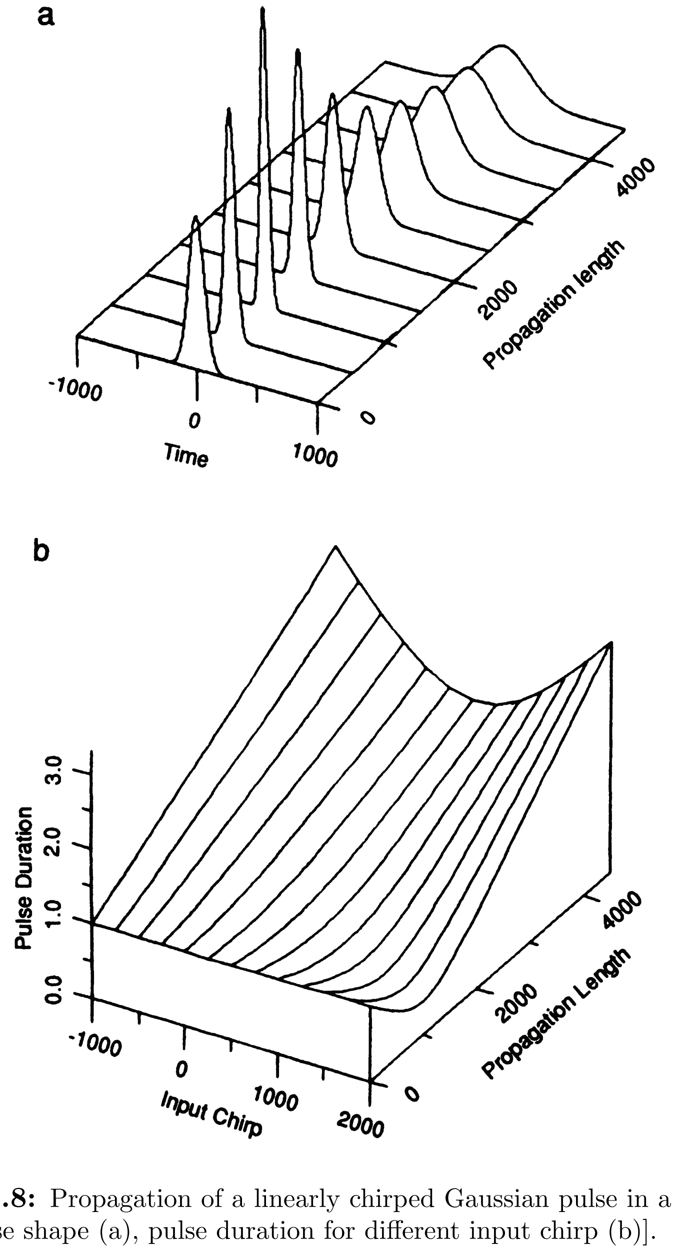

Figure 1.8: Propagation of a linearly chirped Gaussian pulse in a medium with GVD [pulse shape (a), pulse duration for different input chirp (b)].

For BK7 glass at 620 nm, $k''_\ell \approx 6.52692 \times 10^{-26}\mathrm{s}^2/\mathrm{m}$, and the GVD as introduced above is about 320 fs per nm spectral width and meter propagation length.

1.2.5 Complex dielectric constant

In general, the dielectric constant, which was introduced in Eq. (1.73) as a real quantity, is complex. Indeed a closer inspection of Eq. (1.72) shows that the finite memory time of matter requires not only ϵ, χ to be frequency dependent but also that they be complex. The real and imaginary part of $\tilde{\epsilon}$, $\tilde{\chi}$ are not independent of each other but related through a Kramers–Kronig relation. The consideration of a real $\epsilon(\Omega)$ is justified as long as we can neglect (linear) losses or gain. This is valid for transparent samples or propagation lengths which are too short for these processes to become essential for the pulse shaping. For completeness we will modify the reduced wave equation (1.94) by taking into account a complex dielectric constant $\tilde{\epsilon}(\Omega)$ represented as

$$\tilde{\epsilon}(\Omega) = \epsilon(\Omega) + i\epsilon_i(\Omega). \tag{1.143}$$

Let us assume $\tilde{\epsilon}(\Omega)$ to be weakly dispersive. The same procedure introduced to derive Eq. (1.94) can be used after inserting the complex dielectric constant $\tilde{\epsilon}$ into the expression of the polarization Eq. (1.85). Now the reduced wave equation becomes

$$\frac{\partial}{\partial z}\tilde{\mathcal{E}}(t,z) - \frac{i}{2}k''\frac{\partial^2}{\partial t^2}\tilde{\mathcal{E}}(t,z) = \kappa_1\tilde{\mathcal{E}}(t,z) + i\kappa_2\frac{\partial}{\partial t}\tilde{\mathcal{E}}(t,z) + \kappa_3\frac{\partial^2}{\partial t^2}\tilde{\mathcal{E}}(t,z) \tag{1.144}$$

where

$$\kappa_1 = \frac{\omega_\ell}{2}\eta_0\epsilon_i(\omega_\ell) \tag{1.145}$$

$$\kappa_2 = \frac{1}{2}\eta_0\left[2\epsilon_i(\omega_\ell) + \omega_\ell\frac{d}{d\Omega}\epsilon_i(\Omega)\Big|_{\omega_\ell}\right] \tag{1.146}$$

$$\kappa_3 = \frac{1}{4\omega_\ell}\eta_0\left[2\epsilon_i(\omega_\ell) + 4\omega_\ell\frac{d}{d\Omega}\epsilon_i(\Omega)\Big|_{\omega_\ell} + \omega_\ell^2\frac{d^2}{d\Omega^2}\epsilon_i(\Omega)\Big|_{\omega_\ell}\right]. \tag{1.147}$$

In the above expressions, $\eta_0 = \sqrt{\mu_0/\epsilon_0} \approx 377\ \Omega\mathrm{ms}$ is the characteristic impedance of vacuum. For zero–GVD, and neglecting the two last terms in the right-hand side of Eq. (1.144), the pulse evolution with propagation distance z is described by

$$\frac{\partial}{\partial z}\tilde{\mathcal{E}}(t,z) - \kappa_1\tilde{\mathcal{E}}(t,z) = 0 \tag{1.148}$$

which has the solution

$$\tilde{\mathcal{E}}(t,z) = \tilde{\mathcal{E}}(t,0)e^{\kappa_1 z}. \tag{1.149}$$

The pulse experiences losses or gain depending on the sign of κ_1 and does not change its shape. Equation (1.149) states simply the Lambert-Beer law of linear optics.

An interesting situation is that in which there would be neither gain nor loss at the pulse carrier frequency, i.e., $\epsilon_i(\omega_\ell) = 0$ and $\frac{d}{d\Omega}\epsilon_i(\Omega)\Big|_{\omega_\ell} \neq 0$, which could occur between an absorption and amplification line. Neglecting the terms with the second temporal derivative of $\tilde{\mathcal{E}}$, the propagation problem is governed by the equation

$$\frac{\partial}{\partial z}\tilde{\mathcal{E}}(t,z) - i\kappa_2 \frac{\partial}{\partial t}\tilde{\mathcal{E}}(t,z) = 0. \tag{1.150}$$

The solution of this equation is simply

$$\tilde{\mathcal{E}}(t,z) = \tilde{\mathcal{E}}(t + i\kappa_2 z, 0). \tag{1.151}$$

To get an intuitive picture on what happens with the pulse according to Eq. (1.151), let us choose an unchirped Gaussian pulse $\tilde{\mathcal{E}}(t,0)$ [see Eq. (1.33] for $a = 0$), entering the sample at $z = 0$. From Eq. (1.151) we find:

$$\tilde{\mathcal{E}}(t,z) = \tilde{\mathcal{E}}(t,0)\exp\left[\kappa_2^2(z/\tau_G)^2\right]\exp\left[-i2\kappa_2 tz/\tau_G^2\right]. \tag{1.152}$$

The pulse is amplified, and simultaneously its center frequency is shifted with propagation distance. The latter shift is due to the amplification of one part of the pulse spectrum (the high (low) – frequency part if $\kappa_2 < (>)0$) while the other part is absorbed. The result is a continuous shift of the pulse spectrum in the corresponding direction and a net gain while the pulse shape is preserved.

In the beginning of this section we mentioned that there is always an imaginary contribution of the dielectric constant leading to gain or loss. The question arises whether a wave equation such as Eq. (1.94), where only the real part of $\tilde{\epsilon}$ was considered, is of any practical relevance for describing pulse propagation through matter. The answer is yes, because in (almost) transparent regions the pulse change due to dispersion can be much larger than the change caused by losses. An impressive manifestation of this fact is pulse propagation through optical fibers. High-quality fibers made from fused silica can exhibit damping constants as low as 1 dB/km at wavelengths near 1 μm, where the GVD term is found to be $k'' \approx 75$ ps^2/km, see for example [20]. Consequently, a 100 fs pulse launched into a 10 m fiber loses just about 2% of its energy while it broadens by about a factor of 150. To illustrate the physics underlying the striking difference between the action of damping and dispersion, let us consider a dielectric constant $\tilde{\epsilon}(\Omega)$ originating from a single absorption line.

We will use the simple model of a classical harmonic oscillator consisting of an electron bound to a nucleus to calculate the dispersion and absorption of that line. The equation of motion of the electron is:

$$\frac{d^2r}{dt^2} + \omega_0^2 r + \frac{1}{T_c}\frac{dr}{dt} = \frac{e}{m_e}E, \tag{1.153}$$

where $\omega_0 = \sqrt{C/m_e}$ (C being the "spring constant") is the resonance frequency, m_e the electron mass, e its charge, and $1/T_c$ the damping constant. Assuming an electric field of the form $E = (1/2)\tilde{\mathcal{E}}_0 \exp(i\Omega t)$, one finds the polarization $P = N_0 e r$ (N_0 being the number of oscillators (dipoles) per unit volume):

$$P(\Omega) = \frac{N_0 e^2}{m_e}\frac{E}{\omega_0^2 - \Omega^2 + i\Omega/T_c} \tag{1.154}$$

Using the general relation between polarization and electric field $P = \epsilon_0 \chi E$ we obtain an expression for the complex susceptibility:

$$\chi(\Omega) = \frac{N_0^2 e^2}{\epsilon_0 m_e}\frac{1}{\omega_0^2 - \Omega^2 + i\Omega/T_c} \tag{1.155}$$

The real and imaginary parts of the susceptibility χ can be calculated:

$$\chi_r = \frac{N_0 e^2}{\epsilon_0 m_e}\frac{(\omega_0^2 - \Omega^2)}{(\omega_0^2 - \Omega^2)^2 + \Omega^2/T_c^2} \approx \frac{N_0 e^2 T_2}{2 m_e \epsilon_0 \omega_0}\frac{\Delta\omega T_2}{1 + \Delta\omega^2 T_2^2} \tag{1.156}$$

$$\chi_i = -\frac{N_0 e^2}{\epsilon_0 m_e}\frac{(\Omega/T_c)}{(\omega_0^2 - \Omega^2)^2 + \Omega^2/T_c^2} \approx -\frac{N_0 e^2 T_2}{2 m_e \epsilon_0 \omega_0}\frac{1}{1 + \Delta\omega^2 T_2^2} \tag{1.157}$$

The second term of each equation above corresponds to the approximation of small detuning $\Delta\omega = \omega_0 - \Omega \ll \omega_o$. $1/T_2$ is the linewidth of the Lorentzian absorption line, and $T_2 = 2T_c$ will be assimilated in Chapters **??** and **??** to the phase relaxation time of the oscillators. The real and imaginary parts of the oscillator contribution to the susceptibility are responsible for a frequency dependence of the wave vector. One can write

$$k(\Omega) = \Omega\sqrt{\mu_0 \epsilon_0 \left[1 + \chi(\Omega)\right]} \approx \frac{\Omega}{c}\left[1 + \frac{1}{2}\chi(\Omega)\right] \tag{1.158}$$

For frequencies Ω being sufficiently far from resonance, i.e. $|(\omega_0 - \Omega)T_2)| = |\Delta\omega T_2| \gg 1$, but with $|\omega_\ell - \Omega| \ll \omega_\ell$ (narrow pulse spectrum), the real and imaginary parts of the propagation constant are given by:

$$k_r(\Omega) \simeq \frac{\Omega}{c} + B\frac{\Omega}{\Delta\omega T_2} \tag{1.159}$$

$$k_i(\Omega) \simeq -B\frac{\Omega}{(\Delta\omega T_2)^2}, \tag{1.160}$$

where $B = (N_0 e^2 T_2)/(4\epsilon_0\omega_0 c m_e)$. The group velocity dispersion, responsible for pulse reshaping, is:

$$k''(\Omega) \simeq \frac{2BT_2^2\omega_0}{[\Delta\omega T_2]^3}. \tag{1.161}$$

For small travel distances L the relative change of pulse energy can be estimated from Eq. (1.75) and Eq. (1.20) to be:

$$\Delta\mathcal{W}_{rel} = 1 - \frac{\mathcal{W}(L)}{\mathcal{W}(0)} \approx -2k_i L. \tag{1.162}$$

The relative change of pulse duration due to GVD can be evaluated from Eq. (1.126) and we find:

$$\Delta\tau_{rel} = \frac{\tau_G(L)}{\tau_{G0}} - 1 \approx 2\left(\frac{k_\ell'' L}{\tau_{G0}^2}\right)^2. \tag{1.163}$$

To compare both pulse distortions we consider their ratio, using Eqs. (1.160), (1.161), (1.162) and (1.163):

$$\frac{\Delta\tau_{rel}}{\Delta\mathcal{W}_{rel}} = \Delta\mathcal{W}_{rel}\frac{2}{(\Delta\omega T_2)^2}\left(\frac{T_2}{\tau_{G0}}\right)^4. \tag{1.164}$$

At given material parameters and carrier frequency, shorter pulses always lead to a dominant pulse spreading. For $T_2 = 10^{-10}$ s (typical value for a single electronic resonance), and a detuning $\Delta\omega T_2 = 10^4$, we find for example:

$$\frac{\Delta\tau_{rel}}{\Delta\mathcal{W}_{rel}} \approx \Delta\mathcal{W}_{rel}\left(\frac{1200\,\mathrm{fs}}{\tau_{G0}}\right)^4. \tag{1.165}$$

To summarize, a resonant transition of certain spectral width $1/T_2$ influences short pulse (pulse duration < 1 ps) propagation outside resonance mainly due to dispersion. Therefore, the consideration of a transparent material ($\epsilon_i \approx 0$) with a frequency dependent, real dielectric constant $\epsilon(\Omega)$, which was necessary to derive Eq. (1.94), is justified in many practical cases involving ultrashort pulses.

1.3 Interaction of light pulses with linear optical elements

Even though this topic is treated in detail in Chapter **??**, we want to discuss here some general aspects of pulse distortions induced by linear optical elements. These elements comprise typical optical components, such as mirrors, prisms, and gratings, which one usually finds in all optical setups. Here we shall restrict ourselves to the temporal and spectral changes the pulse experiences and shall neglect a possible change of the beam characteristics. A linear optical element of this type can be characterized by a complex optical transfer function

$$\tilde{H}(\Omega) = R(\Omega)e^{-i\Psi(\Omega)} \tag{1.166}$$

that relates the incident field spectrum $\tilde{E}_{in}(\Omega)$ to the field at the sample output $\tilde{E}(\Omega)$

$$\tilde{E}(\Omega) = R(\Omega)e^{-i\Psi(\Omega)}\tilde{E}_{in}(\Omega). \tag{1.167}$$

Here $R(\Omega)$ is the (real) amplitude response and $\Psi(\Omega)$ is the phase response. As can be seen from Eq. (1.167), the influence of $R(\Omega)$ is that of a frequency filter. The phase factor $\Psi(\Omega)$ can be interpreted as the phase delay which a spectral component of frequency Ω experiences. To get an insight of how the phase response affects the light pulse, we assume that $R(\Omega)$ does not change over the pulse spectrum whereas $\Psi(\Omega)$ does. Thus, we obtain for the output field from Eq. (1.167):

$$\tilde{E}(t) = \frac{1}{2\pi}R\int_{-\infty}^{+\infty}\tilde{E}_{in}(\Omega)e^{-i\Psi(\Omega)}e^{i\Omega t}\,d\Omega. \tag{1.168}$$

Replacing $\Psi(\Omega)$ by its Taylor expansion around the carrier frequency ω_ℓ of the incident pulse

$$\Psi(\Omega) = \sum_{n=0}^{\infty} b_n(\Omega - \omega_\ell)^n \tag{1.169}$$

with the expansion coefficients

$$b_n = \frac{1}{n!}\frac{d^n\Psi}{d\Omega^n}\bigg|_{\omega_\ell} \tag{1.170}$$

we obtain for the pulse

$$\begin{aligned}\tilde{E}(t) &= \frac{1}{2}\tilde{\mathcal{E}}(t)e^{i\omega_\ell t}\\ &= \frac{1}{2\pi}Re^{-ib_0}e^{i\omega_\ell t}\int_{-\infty}^{+\infty}\tilde{E}_{in}(\Omega)\end{aligned}$$

$$\times \exp\left(-i\sum_{n=2}^{\infty} b_n(\Omega-\omega_\ell)^n\right) e^{i(\Omega-\omega_\ell)(t-b_1)}\, d\Omega. \quad (1.171)$$

By means of Eq. (1.171) we can easily interpret the effect of the various expansion coefficients b_n. The term e^{-ib_0} is a constant phase shift (phase delay) having no effect on the pulse envelope. A nonvanishing b_1 leads solely to a shift of the pulse on the time axis t; the pulse would obviously keep its position on a time scale $t' = t - b_1$. The term b_1 determines a group delay in a similar manner as the first–order expansion coefficient of the propagation constant k defined a group velocity in Eq. (1.98). The higher–order expansion coefficients produce a nonlinear behavior of the spectral phase which changes the pulse envelope and chirp. The action of the term with $n = 2$, for example, producing a quadratic spectral phase, is analogous to that of GVD in transparent media.

If we decompose the input field spectrum into modulus and phase $\tilde{E}_{in}(\Omega) = |\tilde{E}_{in}(\Omega)|\exp(i\Phi_{in}(\Omega))$, we obtain from Eq. (1.167) for the spectral phase at the output

$$\Phi(\Omega) = \Phi_{in}(\Omega) - \sum_{n=0}^{\infty} b_n(\Omega-\omega_\ell)^n. \quad (1.172)$$

It is interesting to investigate what happens if the linear optical element is chosen to compensate for the phase of the input field. For Taylor coefficients with $n \geq 2$:

$$b_n = \frac{1}{n!}\frac{d^n}{d\Omega^n}\Phi_{in}(\Omega)\bigg|_{\omega_\ell}. \quad (1.173)$$

A closer inspection of Eq. (1.171) shows that when Eq. (1.173) is satisfied, all spectral components are in phase for $t - b_1 = 0$, leading to a pulse with maximum peak intensity, as was discussed in previous sections. We will come back to this important point when discussing pulse compression. We want to point out the formal analogy between the solution of the linear wave equation (1.75) and Eq.(1.167) for $R(\Omega) = 1$ and $\Psi(\Omega) = k(\Omega)z$. This analogy expresses the fact that a dispersive transmission object is just one example of a linear element. In this case we obtain for the spectrum of the complex envelope

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{\mathcal{E}}_{in}(\Omega, 0)\exp\left[-i\sum_{n=0}^{\infty}\frac{1}{n!}k_\ell^{(n)}(\Omega-\omega_\ell)^n z\right] \quad (1.174)$$

where $k_\ell^{(n)} = (d^n/d\Omega^n)k(\Omega)|_{\omega_\ell}$.

Next let us consider a sequence of m optical elements. The resulting transfer function is given by the product of the individual contributions $\tilde{H}_j(\Omega)$

$$\tilde{H}(\Omega) = \prod_{j=1}^{m} \tilde{H}_j(\Omega) = \left(\prod_{j=1}^{m} R_j(\Omega)\right) \exp\left[-i\sum_{j=1}^{m} \Psi_j(\Omega)\right] \tag{1.175}$$

which means an addition of the phase responses in the exponent. Subsequently, by a suitable choice of elements, one can reach a zero-phase response so that the action of the device is through the amplitude response only. In particular, the quadratic phase response of an element (e.g., dispersive glass path) leading to pulse broadening can be compensated with an element having an equal phase response of opposite sign (e.g., grating pair) which automatically would re-compress the pulse to its original duration. Such methods are of great importance for the handling of ultrashort light pulses. Corresponding elements will be discussed in Chapter **??**.

1.4 Generation of phase modulation

At this point let us briefly discuss essential physical mechanisms to produce a time dependent phase of the pulse, i.e., a chirped light pulse. Processes resulting in a phase modulation can be divided into those that increase the pulse spectral width and those that leave the spectrum unchanged. The latter can be attributed to the action of linear optical processes. Any transparent linear medium, or spectrally "flat" reflector, can change the phase of a pulse, without affecting its spectral amplitude. The action of these elements is most easily analyzed in the frequency domain. As we have seen in the previous section, the phase modulation results from the different phase delays which different spectral components experience upon interaction. The result for an initially bandwidth-limited pulse, in the time domain, is a temporally broadened pulse with a certain frequency distribution across the envelope, such that the spectral amplitude profile remains unchanged. For an element to act in this manner its phase response $\Psi(\Omega)$ must have non-zero derivatives of at least second order as explained in the previous section.

A phase modulation that leads to a spectral broadening is most easily discussed in the time domain. Let us assume that the action of a corresponding optical element on an unchirped input pulse can be formally written as:

$$\tilde{E}(t) = T(t)e^{i\Phi(t)}\tilde{E}_{in}(t) \tag{1.176}$$

where T and Φ define a time dependent amplitude and phase response, respectively. For our simplified discussion here let us further assume that T = const., leaving the pulse envelope unaffected. Since the output pulse has an additional phase modulation $\Phi(t)$ its spectrum must have broadened during the interaction. If the pulse under consideration is responsible for the time dependence of Φ, then we call the process self-phase modulation. If additional pulses cause the temporal change of the optical properties we will refer to it as cross-phase modulation. Often, phase modulation occurs through a temporal variation of the index of refraction n of a medium during the passage of the pulse. For a medium of length d the corresponding phase is:

$$\Phi(t) = -k(t)d = -\frac{2\pi}{\lambda}n(t)d. \tag{1.177}$$

In later chapters we will discuss in detail several nonlinear optical interaction schemes with short light pulses that can produce a time dependence of n.

A time dependence of n can also be achieved by applying a voltage pulse at an electro-optic material for example. However, with the view on phase shaping of femtosecond light pulses the requirements for the timing accuracy of the voltage pulse make this technique difficult.

1.5 Beam propagation

1.5.1 General

So far we have considered light pulses propagating as plane waves, which allowed us to describe the time varying field with only one spatial coordinate. This simplification implies that the intensity across the beam is constant and, moreover, that the beam diameter is infinitely large. Both features hardly fit what we know from laser beams. Despite the fact that both features do not match the real world, such a description has been successfully applied for many practical applications and will be used in this book whenever possible. This simplified treatment is justified if the processes under consideration either do not influence the transverse beam profile (e.g., sufficiently short sample length) or allow one to discuss the change of beam profile and pulse envelope as if they occur independently from each other. The general case, where both dependencies mix, is often more complicated and, frequently, requires extensive numerical treatment. Here we will discuss solely the situation where the change of such pulse characteristics as duration, chirp, and bandwidth can be separated from the change of the beam profile. Again we restrict ourselves to a linearly polarized field which now has to be considered in its complete spatial dependence. Assuming a propagation in the z-direction, we can write the field in the form:

$$E = E(x, y, z, t) = \frac{1}{2}\tilde{u}(x, y, z)\tilde{\mathcal{E}}(t)e^{i(\omega t - k_\ell z)} + c.c.. \tag{1.178}$$

In the definition (1.178) the scalar $\tilde{u}(x, y, z)$ is to describe the transverse beam profile and $\tilde{\mathcal{E}}(t, z)$ is the slowly varying complex envelope introduced in Eq. (1.83). Note that the rapid z-dependence of E is contained in the exponential function. Subsequently, $\tilde{u}$ is assumed to vary slowly with z. Under these conditions the insertion of Eq. (1.178) into the wave equation (1.68) yields after separation of the time dependent part in paraxial approximation [11]:

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2ik_\ell \frac{\partial}{\partial z}\right)\tilde{u}(x, y, z) = 0,} \tag{1.179}$$

which is usually solved by taking the Fourier transform along the space coordinates x and y, yielding:

$$\left[\frac{\partial}{\partial z} - \frac{i}{2k_\ell}\left(k_x^2 + k_y^2\right)\right]\tilde{u}(k_x, k_y, z) = 0, \tag{1.180}$$

where k_x and k_y are the Fourier variables (spatial frequencies, wave numbers). This equation can be integrated, to yield the integral form of Fresnel equation:

$$\tilde{u}(k_x, k_y, z) = \tilde{u}(k_x, k_y, 0)e^{\frac{i}{2k_\ell}\left(k_x^2 + k_y^2\right)z}. \tag{1.181}$$

Paraxial approximation means that the transverse beam dimensions remain sufficiently small compared with typical travel distances of interest. An important particular solution of the wave equation within the paraxial approximation is the Gaussian beam (see, e.g., [11]), which can be written in the form:

$$\tilde{u}(x, y, z) = \frac{u_0}{\sqrt{1 + z^2/\rho_0^2}} e^{-i\Theta(z)} e^{-ik_\ell(x^2+y^2)/2R(z)} e^{-(x^2+y^2)/w^2(z)}. \tag{1.182}$$

where

$$R(z) = z + \rho_0^2/z \tag{1.183}$$

$$w(z) = w_0\sqrt{1 + z^2/\rho_0^2} \tag{1.184}$$

$$\Theta(z) = \arctan(z/\rho_0) \tag{1.185}$$

$$\rho_0 = \frac{n\pi w_0^2}{\lambda}. \tag{1.186}$$

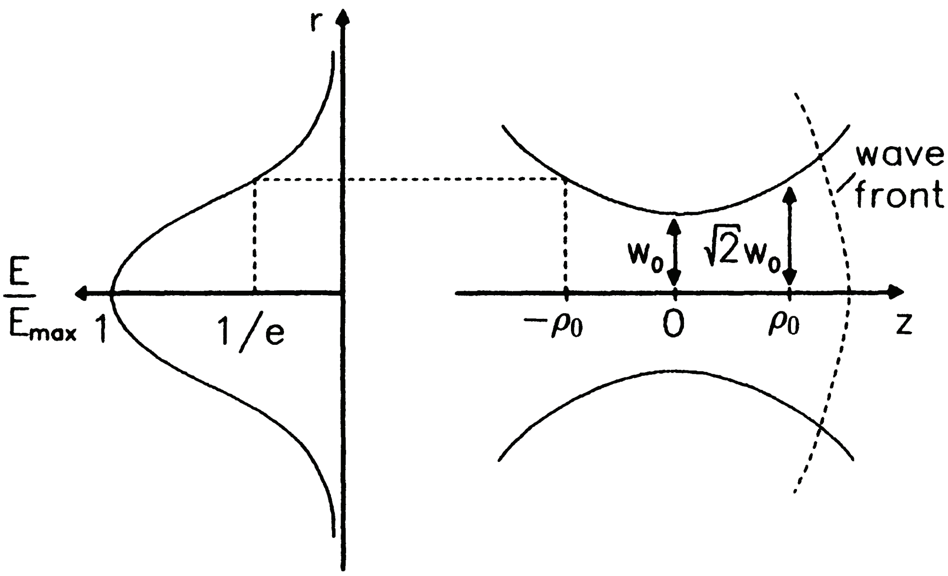

Figure 1.9: Parameters of Gaussian beams

Sometimes it is convenient to write Eq. (1.182) as

$$\tilde{u}(x, y, z) = \frac{u_0}{\sqrt{1 + z^2/\rho_0^2}} e^{-i\Theta(z)} e^{-ik_\ell(x^2+y^2)/2\tilde{q}(z)} \tag{1.187}$$

where $\tilde{q}(z)$ is the complex beam parameter which is defined by:

$$\frac{1}{\tilde{q}(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w^2(z)} = \frac{1}{\tilde{q}(0) + z}. \tag{1.188}$$

Optical beams described by Eq. (1.182) exhibit a Gaussian intensity profile transverse to the propagation direction with $w(z)$ as a measure of the beam diameter, as sketched in Fig. 1.9. The origin of the z-axis ($z = 0$) is chosen to be the position of the beam waist $w_0 = w(z = 0)$. The radius of curvature of planes of constant phase is $R(z)$, Its value is infinity at the beam waist (plane phase front)[3] and at $z = \infty$. The length ρ_0 is called the Rayleigh range; $2\rho_0$ being the confocal parameter. For $-\rho_0 \geq z \leq \rho_0$, the beam size is within the limits $w_0 \leq w \leq \sqrt{2}w_0$. Given the amplitude u_0 at a given

[3]The phase term $\Theta(z)$ in Eq. (1.182) takes on a constant value and need not be considered for $z \gg \rho_0$. A Gaussian beam at the position of its waist must not be confused with a plane wave.

beam waist and wavelength λ, the field at an arbitrary position (x, y, z) is completely predictable by means of Eqs. (1.182) through (1.186).

Instead of using the differential equation (1.179), one can equivalently describe the field propagation by an integral equation. The basic approach is to start with Huygens' principle, and apply the Fresnel approximation assuming paraxial wave propagation [11]. Assuming that the field distribution (or beam profile) $\tilde{u}(x', y', z') = \tilde{u}_0(x', y')$ is known at a plane $z' =$ const.; the field distribution $\tilde{u}(x, y, z)$ at a plane $z = z' + L$ is given by:

$$\tilde{u}(x, y, z) = \frac{i e^{i k_\ell L}}{\lambda L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}_0(x', y') e^{-i k_\ell [(x'-x)^2 + (y'-y)^2]/(2L)} dx' dy'. \tag{1.189}$$

Note that both ways of describing the field variation due to diffraction are equivalent. One can easily show that the field (1.189) is a convolution of $\tilde{u}(x, y, 0)$ and $\exp[-ik(x^2 + y^2)/(2L)]$.

1.6 Analogy between pulse and beam propagation

1.6.1 Time analogy of the paraxial (Fresnel) approximation

Comparing the paraxial wave equation (1.179) and the reduced wave equation (1.94) describing pulse propagation through a GVD medium we notice an interesting correspondence. Both equations are of similar structure. In terms of the reduced wave equation the transverse space coordinates x, y in Eq. (1.179) seem to play the role of the time variable. This space-time analogy suggests the possibility of translating simply the effects related to dispersion into beam propagation properties. For instance, we may compare the temporal broadening of an unchirped pulse due to dispersion with the change of beam size due to diffraction. In this sense free-space propagation plays a similar role for the beam characteristics as a GVD medium does for the pulse envelope. To illustrate this in more detail let us start with Eq. (1.181), and, for simplicity, restrict ourselves to one dimension. The field spectrum at a distance z is:

$$\tilde{u}(k_x, z) = \tilde{u}(k_x, z = 0) e^{i k_x^2 z/(2k_\ell)}, \tag{1.190}$$

which has a inverse Fourier transform the convolution product:

$$\tilde{u}(x, z) \propto \mathcal{F}^{-1} \left\{ \tilde{u}_0(k_x) e^{i z k_x^2/(2k_\ell)} \right\} = \int_{-\infty}^{\infty} \tilde{u}(x_0, 0) e^{-i \frac{k_\ell}{2z} (x - x_0)^2} dx_0 \tag{1.191}$$

which is the well known Fresnel integral. Let us next recall Eq. (1.110), approximated to second order, which states that the spectral envelope after propagation through a thickness z of a linear transparent material is given by:

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{\mathcal{E}}(\Omega, 0) e^{-\frac{i}{2} k_\ell'' \Omega^2 z} \tag{1.192}$$

A comparison with Eq. (1.191) clearly shows the similarity between the diffraction and the dispersion problem. The exponential phase factor $k_x^2 z/(2k)$ which describes transverse beam diffraction in space, corresponds to the exponential phase factor $-k"\Omega^2 z/2$ which describes pulse dispersion in time. The time-fresnel integral (1.192) can also be written as a convolution in the time domain:

$$\tilde{\mathcal{E}}(t, z) \propto= \int_{-\infty}^{\infty} \tilde{\mathcal{E}}(t_0, 0) e^{-i \frac{(t-t_0)^2}{2k_\ell" z}} dt_0 \tag{1.193}$$

Since Eq. (1.190) corresponded to the paraxial approximation, the analogy can be carried over to successive subsets of that approximation. It will thus apply also to Gaussian optics, and the time equivalent of the Fraunhofer and geometric approximations, asa will be shown in subsequent sections.

1.6.2 Time analogy of the Fraunhofer approximation

The Fraunhofer approximation can be derived from the Fresnel integral by inserting the approximation:

$$\frac{k_\ell}{2z}(x - x_0)^2 \approx -ik_\ell \frac{x}{z} x_0 = -ik_x x_0, \tag{1.194}$$

in Eq. (1.191), yielding:

$$\tilde{u}(x, z) \propto \int_{-\infty}^{\infty} \tilde{u}(x_0, 0) e^{-ik_x x_0} dx_0, \tag{1.195}$$

where k_x is the projection of the k_ℓ vector in the plane of observation of the diffraction pattern, and takes the following forms:

- Image plane at finite distance: $k_x = k_\ell(x/z)$, where x is the coordinate in the observation plane
- Image plane at infinity: $k_x = k_\ell \theta_x$, where θ_x is the angle of observation
- Image plane at the focal plane of a lens: $k_x = k_\ell(x/f)$ where f is the focal distance of the lens.

Equivalently, a time Fraunhofer approximation can be derived from the time Fresnel integral by inserting the approximation:

$$\frac{k_\ell}{2z}(t - t_0)^2 \approx -ik_\ell \frac{x}{z} x_0 = -ik_x x_0, \tag{1.196}$$

in Eq. (1.191), yielding:

$$\tilde{\mathcal{E}}(t,z) \propto \int_{-\infty}^{\infty} \tilde{\mathcal{E}}(t_0,0) e^{-i\frac{t}{k_\ell" z}t_0} dt_0, \qquad (1.197)$$

which is the Fourier transform of the initial field, calculated at a frequency $\Omega_1 = t/(k_\ell" z)$. Of the three different conditions in space cited above (Image plane at finite, infinite distance or at the focus of a lens, two conditions subsist:

- The observation distance z is after a simple linear dispersion, and $\Omega_1 = t/(k_\ell" z)$
- The observation distance is at the focal distance of a time lens, and $\Omega_1 = t/f_T$ where $f_T = 1/\ddot{\phi}$ is the focal distance of the "time lens" created by an applied phase modulation $\ddot{\phi}$ (cf. Section 1.6.3)[4].

The physical meaning of these analogies are that, after propagation of long distance in a dispersive medium, the temporal variation of any signal is replaced by its Fourier transform.

1.6.3 Geometric optics in time

In geometric optics, one considers sources to be δ-functions, and rays to propagate in straight lines. We will start first with imaging, showing that the simple lens equation applied also to temporal optics. Next we will show how one of the most powerful tool in spatial optics — the propagation matrix — can be used in time domain and applied to the calculation of laser cavities.

Analogy between spatial and temporal imaging

The analogy between pulse and beam propagation was applied to establish a time–domain analog of an optical imaging system by Kolner and Nazarathy [21]. Optical microscopy, for example, serves to magnify tiny structures so that they can be observed by a (relatively) low–resolution system such as our eyes. The idea of the "time lens" is to magnify ultrafast (fs) transients so that they can be resolved, for example, by a relatively slow oscilloscope. Of course, the opposite direction is also possible, which would lead to data compression in space or time. While Table 1.2 illustrates the space–time duality for free–space propagation we now need to look for

[4] The distance z of observation is then $z = f_T/k_\ell"$, where $k_\ell"$ is the dispersion of the medium following the time lens

devices resembling imaging elements such as lenses. From Fourier optics it is known that a lens introduces a quadratic phase factor, thus transforming a (Fourier–limited) input beam (parallel beam) into a spatially chirped (focused) beam. The "time equivalent" lens is a quadratic phase modulator. Quadratic dispersion through a medium with GVD is the temporal analogue of diffraction. Let us consider the lens arrangement of Fig. 1.10, in which the light from an object — represented by the field envelope $\mathcal{E}(r)$ —, at a distance d_1 from the lens, is imaged on a screen at a distance d_2 from the lens. The real image is produced on the screen if the distance and focal distance of the lens satisfy the lens formula:

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f} \tag{1.198}$$

With some approximations, one can derive the time–domain equivalent of the Gaussian lens formula [21], for an optical system [Fig. 1.10 (b)] in which the initial signal $\tilde{\mathcal{E}}(t)$ is propagated for a distance d_1 through a dispersive medium characterized by a wave vector k_2, is given a quadratic phase modulation by a "time lens", and propagates for a distance d_2 through a medium of wave vector k_2:

$$\left(d_1 \frac{d^2 k_1}{d\Omega^2}\right)^{-1} + \left(d_2 \frac{d^2 k_2}{d\Omega^2}\right)^{-1} = \left(f_T/\omega_0\right)^{-1}. \tag{1.199}$$

In this "temporal lens formula", $d_{1,2}(d^2 k_{1,2}/d\Omega^2)$ are the dispersion characteristics of the object and image side, respectively, and $\omega_0/f_T = \partial^2\phi/\partial t^2$ is the parameter of the quadratic phase modulation impressed by the modulator. As in optical imaging, to achieve large magnification with practical devices, short focal lengths are desired. For time imaging this translates into a short focal time f_T which in turn requires a suitably large phase modulation.

Note that the real image of an object can only be recognized on a screen located at a specific distance from the lens, i.e., in the image plane. At any other distance the intensity distribution visible on a screen does usually not resemble the object, because of diffraction. Likewise, the dispersive element broadens each individual pulse (if we assume zero input chirp). It is only after the time lens and a suitably designed second dispersive element that a "pulse train" with the same contrast as the input (but stretched or compressed) emerges.

One possible approach to create a large phase modulation is cross-phase modulation, in which a properly shaped powerful "pump" pulse creates a

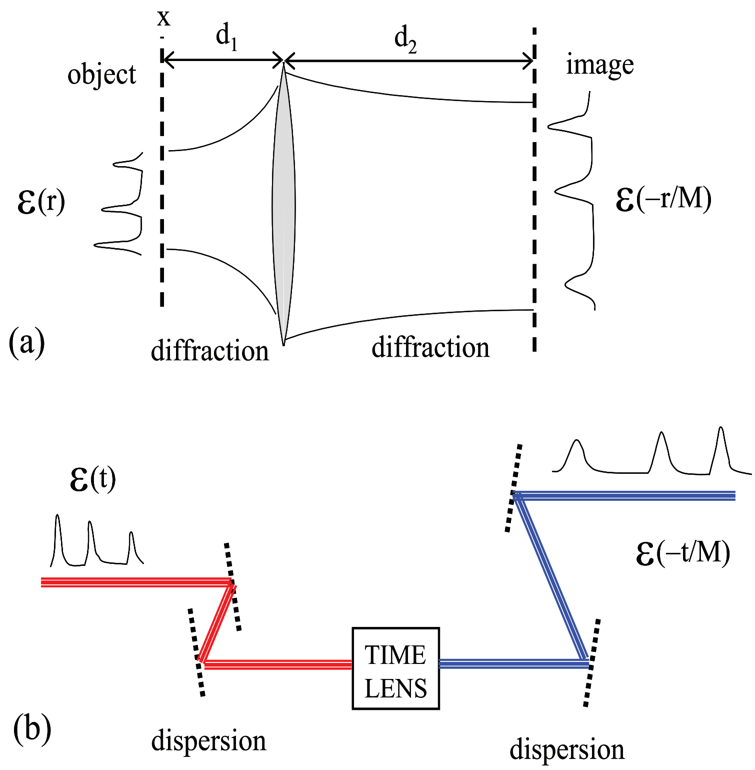

Figure 1.10: Space–time analogy of imaging. (a) Spatial imaging configuration. The "object" is a graphic representation of a succession of a three pulse sequence. The "real image" shows a magnified, inverted picture. (b) The temporal imaging configuration. A pair of gratings on either side of the time lens represents a dispersive length characterized by $d^2k/d\Omega^2$, see also Chapter **??**. The object is a three pulse sequence. The "image" is a reversed, expanded three pulse sequence. Possible time lenses are explained in the text. (Adapted from [22].)

large index sweep (quadratic with time) in the material of the "time lens". Another approach is to use sum or difference frequency generation to impart the linear chirp of one pulse into the pulse to be "imaged". The linear chirp can be obtained by propagating of a strong pulse through a fiber. A detailed review of this "parametric temporal imaging" can be found in refs. [23, 22]. The time-equivalent of a long propagation distance (large diffraction) is a large dispersion, which can be obtained with a pair of gratings, see Chapter **??**. Note that in a standard magnifying optical system with a single lens, the real image is inverted with respect to the object. The same applies to the temporal imaging: the successive pulses appear in reverse order in the image.

ABCD matrix

Review of ABCD matrix in space An ABCD matrix [24] is a ray transfer matrix which describes the effect of an optical element on a laser beam. It can be used both in geometrical optics and for propagating Gaussian beams. The paraxial approximation is always required for ABCD matrix calculations. Tracing of a light path through an optical system can then be performed by multiplying an element matrix by a vector representing the light ray:

$$\begin{pmatrix} y_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix} \tag{1.200}$$

where y and α refer to transverse displacement and offset angle from an optical axis respectively. The subscripts '1' and '2' denote the coordinates before and after an optical element. For example, a thin lens with focal length f has the following ABCD matrix:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \tag{1.201}$$

and propagation through free space over a distance d is associated with the matrix:

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{1.202}$$

transition from space to time Geometric optics can be seen as an approximation of Gaussian propagation, where the propagation distance is much larger than the Rayleigh range. Therefore, the propagation in a linear medium is in a straight line making an angle α with the optics axis:

$$w = w_0\sqrt{1 + \left(\frac{z}{\rho_0}\right)^2} \approx \frac{w_0}{\rho_0}z = \alpha z = \frac{2}{k_\ell w_0}z. \tag{1.203}$$

The same approximation can be made in the time domain to define the "optical inclination" α:

$$\tau = \tau_0\sqrt{1 + \left(\frac{z}{L_d}\right)^2} \approx \frac{\tau_0}{L_d}z = \alpha_T z = \frac{2k"_\ell}{\tau_{G0}}z. \tag{1.204}$$

Note that in contrast to the space domain where α is dimensionless, α_T has the dimension of the inverse of a velocity. The generic operation of Eq. (1.200) has its time domain equivalent:

$$\begin{pmatrix} T_2 \\ \alpha_{T,2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ \alpha_{T,1} \end{pmatrix} \tag{1.205}$$

where T, the correspondent of the transverse displacement y, is a temporal position. In the time equivalent of the propagation matrix (1.202), the distance d is replaced by $k"_\ell d$, while the time equivalent of the lens matrix (1.201) has the element $-1/f$ replaced by an imposed chirp $\ddot{\Phi}$ which, for instance, in the case of Kerr modulation, is equal to:

$$\ddot{\Phi} = \frac{2\pi \ell_{Kerr}}{\lambda} n_2 \frac{I}{\tau_G^2}, \tag{1.206}$$

where ℓ_{Kerr} is the length of the nonlinear medium characterized by an intensity dependent index $n_2 I$. The matrix representation of the imaging problem of Fig. 1.10, in space, is:

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix}, \tag{1.207}$$

while the time correspondent is:

$$\begin{pmatrix} T' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & k"_2 d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\ddot{\Phi} & 1 \end{pmatrix} \begin{pmatrix} 1 & k"_1 d_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T \\ \alpha \end{pmatrix} \tag{1.208}$$

The various level of approximation of space-time analogy are summarized in Fig. **??**.

1.6.4 Gaussian pulses as analogue of Gaussian beams

As we have seen in the previous section the quadratic phase factor in Eq. (1.192) broadens an unchirped input pulse and leads to a (linear) frequency sweep across the pulse (chirp) while the pulse spectrum remains unchanged. In an analogous manner we can interpret Eq. (1.191) for the beam profile. A "bandwidth-limited" Gaussian beam means a beam without phase variation across the beam, which, in terms of Eq. (1.182), requires a radius of curvature of the phase front $R = \infty$. Thus, a Gaussian beam is "bandwidth-limited" at its waist where it takes on its minimum possible size (at a given spatial frequency spectrum). Multiplication with a quadratic phase factor

Gaussian pulse	**Gaussian beam**
bandwidth-limited pulse at $z = 0$ (unchirped pulse) $$\begin{aligned} \tilde{\mathcal{E}}_0(t) &\propto e^{-(t/\tau_{G0})^2} \\ \tilde{\mathcal{E}}_0(\Omega) &\propto e^{-(\tau_{G0}\Omega/2)^2} \end{aligned}$$	beam waist at $z = 0$ (plane phase fronts) $$\begin{aligned} \tilde{u}_0(x) &\propto e^{-(x/w_0)^2} \\ \tilde{u}_0(k_x) &\propto e^{-(k_x w_0/2)^2} \end{aligned}$$
Propagation through a medium of length L (dispersion) $$\begin{aligned} \tilde{\mathcal{E}}(\Omega, L) &\propto \exp\left[-\left(\frac{\tau_{G0}\Omega}{2}\right)^2 - i\frac{k_\ell'' L\Omega^2}{2}\right] \\ \tilde{\mathcal{E}}(t, L) &\propto \exp\left[-(1+i\bar{a})\left(\frac{t}{\tau_G}\right)^2\right] \\ &\propto \exp\left[i\omega_\ell \frac{t^2}{2\tilde{p}(L)}\right] \\ \bar{a} &= L/L_d \\ \tau_G(L) &= \tau_{G0}\sqrt{1+\bar{a}^2} \end{aligned}$$	Free space propagation over distance L (diffraction) $$\begin{aligned} \tilde{u}(k_x, L) &\propto \exp\left[-\left(\frac{w_0 k_x}{2}\right)^2 + i\frac{L{k_x}^2}{2k_\ell}\right] \\ \tilde{u}(x, L) &\propto \exp\left[-(1+i\bar{b})\left(\frac{x}{w}\right)^2\right] \\ &\propto \exp\left[-ik_\ell \frac{x^2}{2\tilde{q}(L)}\right] \\ \bar{b} &= L/\rho_0 \\ w(L) &= w_0\sqrt{1+\bar{b}^2} \end{aligned}$$
Chirp coefficient (slope) $$\ddot{\varphi} = \frac{2\bar{a}}{1+\bar{a}^2}\,\frac{1}{\tau_{G0}^2}$$	Wavefront curvature $$\frac{1}{R} = \frac{\bar{b}}{1+\bar{b}^2}\,\frac{1}{\rho_0}$$
Characteristic (dispersion) length $$L_d = \frac{\tau_{G0}^2}{2k_\ell''}$$	Characteristic (Rayleigh) length $$\rho_0 = \frac{n\pi w_0^2}{\lambda_\ell} = \frac{k_\ell w_0^2}{2}$$
Complex pulse parameter $$\frac{1}{\tilde{p}(L)} = \ddot{\varphi}(L) + \frac{2i}{\tau_G^2(L)}$$	Complex beam parameter $$\frac{1}{\tilde{q}(L)} = \frac{1}{R(L)} + \frac{i\lambda_\ell}{\pi w^2(L)}$$

Table 1.2: Comparison of dispersion and diffraction

SPACE **TIME**

$$\left(\Delta_{tr} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{E}e^{i(\omega t - kz} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} - 2ik\frac{\partial}{\partial z}\right)\mathcal{E}(x,z) = 0 \qquad \text{F. T.} \qquad \left[\frac{\partial}{\partial z} + k(\Omega)\right]\mathcal{E}(\Omega,z) = 0$$

$$\mathcal{E}(k_x,z) = \mathcal{E}(k_x,0)e^{\frac{i}{2k_\ell}k_x^2 z} \qquad \mathcal{E}(\Omega,z) = \mathcal{E}(\Omega,0)e^{\frac{-ik_\ell"}{2}\Omega^2 z}$$

$$\boxed{\frac{k_x^2}{2k_\ell} \longleftrightarrow \frac{-k_\ell"\Omega^2}{2}}$$

$$\tilde{u}(x,z) \propto \int_{-\infty}^{\infty}\tilde{u}(x_0,0) \qquad \text{Fraunf.} \qquad \tilde{\mathcal{E}}(t,z) \propto \int_{-\infty}^{\infty}\tilde{\mathcal{E}}(t_0,0)e^{-i\frac{t}{k_\ell" z}t_0}dt_0$$

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \qquad \text{Displacement matrix} \qquad \begin{pmatrix} 1 & k"_\ell d \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad \text{Lens matrix} \qquad \begin{pmatrix} 1 & 0 \\ -\ddot{\Phi} & 1 \end{pmatrix}$$

Figure 1.11: Space-time equivalence, starting from the Fourier transform of Maxwell's equation in space (left) and in time (right). F.T. indicates Fourier transform. Fraunh. is the Fraunhofer approximation.

to describe the beam propagation, cf. Eq. (1.191), leads to beam broadening and "chirp." The latter simply accounts for a finite phase front curvature. Roughly speaking, the spatial frequency components which are not needed to form the broadened beam profile are responsible for the beam divergence. Table 1.2 summarizes our discussion comparing the characteristics of Gaussian beam and pulse propagation.

1.6.5 Time-space analogy applied to cavity calculations

"p" complex parameter, time correspondent of the spatial "q" parameter

A convenient quantity, labeled the q parameter, has been defined for Gaussian beams. It concatenates the information on the beam radius w and the radius of curvature R in a single complex quantity defined by:

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi w^2} \tag{1.209}$$

It has been observed that the modification of the q parameter by an optical element can be expressed in terms of the elements of the $ABCD$ matrix:

$$\frac{1}{q_2} = \frac{C + D/q_1}{A + B/q_1} \tag{1.210}$$

where q_1 and q_2 represent the value of the q parameter before and after the optical element, respectively. Equation (1.210) is often represented in the form:

$$\begin{pmatrix} q_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ 1 \end{pmatrix}. \tag{1.211}$$

For equivalence with Eq. (1.210), a re-normalization of the 'q' vector is needed after the matrix multiplication.

An example of application of ABCD matrices in space is the study of cavity stability. For a cavity characterized by an ABCD matrix, the evolution of the 'q' parameter over N round trips is given by:

$$\begin{pmatrix} q_N \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} q_{N-1} \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{N-1} \cdot \begin{pmatrix} q_1 \\ 1 \end{pmatrix} \tag{1.212}$$

It can be shown [24] that, for the beam to be trapped in the cavity, there is a stability condition: $-1 \leq \frac{1}{2}(A + D) \leq 1$.

The time equivalent of the q parameter:

$$\frac{1}{p} = \ddot{\varphi} - \frac{2i}{\tau_G^2}, \tag{1.213}$$

where $\ddot{\varphi} = \frac{\partial^2\varphi}{\partial t^2}$ is the second derivative of the phase in the middle of the pulse, and τ_G [remembering that $\tau_p = \sqrt{2\ln 2}\tau_G$] is the Gaussian pulse width. The matrices for dispersion and time lensing as defined in Eq. (1.208)can be applied.

Application to a simple mode-locked cavity – stability criterium

To further clarify the time-space analogy, we consider a simple mode-locked laser cavity as sketched in Fig. 1.12. The time equivalent of the "flat mirror – curved mirror (radius R)" cavity [length L - Fig. 1.12 (a)], is one that

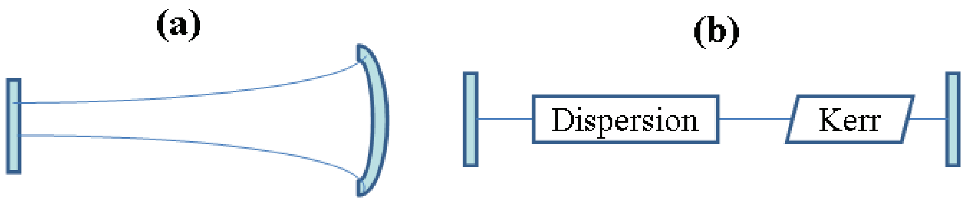

Figure 1.12: Example of s simple laser cavity in space (a) and in time (b). The round-trip matrix start at the flat mirror (before the dispersion in (b).

starts from a bandwidth limited pulse at one end, propagates through the dispersion (k"L) of the cavity, goes through a Kerr self phase modulation ($\ddot{\Phi}$), then dispersion again to the starting point [Fig. 1.12 (a)]. The ABCD matrix for this cavity is, in space:

$$\begin{pmatrix} 1 - \frac{2}{R}L & 2L(1 - \frac{2}{R}L) \\ -4\frac{L}{R} & 1 - \frac{2}{R}L \end{pmatrix} \tag{1.214}$$

and in time:

$$\begin{pmatrix} 1 + 2k''\ell\ddot{\Phi} & 2k''\ell(1 + k''\ell\ddot{\Phi}) \\ 2\ddot{\Phi} & 1 + 2k''\ell\ddot{\Phi} \end{pmatrix} \tag{1.215}$$

Stable operation of the laser requires that the q or p parameter is equal to itself after a round-trip [24] leading to the solution:

$$\frac{1}{p} = \frac{D - A}{2B} \mp \frac{i}{2B}\sqrt{4 - (A + D)^2}. \tag{1.216}$$

For a real solution to exist, the stability criterium is:

$$(A + D)^2 < 4 \tag{1.217}$$

which for the space cavity implies $R \leq \infty$ and $R \geq L$; the latter limit giving the smallest beam waist at the flat mirror (concentric cavity). For the time cavity, the stability criterium is $-2 < \ddot{\Phi}k\text{"}L < 0$, which implies opposite sing for the phase modulation and dispersion. The minimum pulse in the time cavity is given by:

$$\tau_G^2 = 4B\sqrt{4 - (A + D)^2} = 16k\text{"}L(1 + \ddot{\Phi}k\text{"}L)\sqrt{2\ddot{\Phi}k\text{"}L - (\ddot{\Phi}k\text{"}L)^2}. \tag{1.218}$$

The shortest pulse duration is achieved for the "time concentric" cavity with $\ddot{\Phi}k\text{"}L = -2$.

1.7 Numerical modeling of pulse propagation

The generation and application of femtosecond light pulses requires one to study their propagation through linear and nonlinear optical media. Those studies have been undertaken not only to satisfy theorists. They are very necessary to design and optimize experiments, and to save time and money. Because of the complexity of interactions taking place numerical methods have to be used in many cases. From the mathematical point of view it is desirable to develop a numerical model optimized with respect to computer time and accuracy for each experimental situation to be described. In this section we will present a procedure that allows one to study pulse propagation through a variety of materials. This model is optimized neither with respect to computer time nor with respect to accuracy. However, it is very universal and is directly associated with the physics of the problem. Moreover, it has been successfully applied to various situations. Among them are, for instance, pulse propagation through nonlinear optical fibers and amplifiers and pulse evolution in fs lasers. Without going into the numerical details, we will briefly describe the main features of this concept. In the course of the book we will then present various examples.

In the frame of approximations discussed in the section of beam propagation the electric field can be represented as

$$\begin{aligned} E(x,y,z,t) &= \frac{1}{2}\tilde{u}(x,y,z)\tilde{\mathcal{E}}(z,t)e^{i(\omega_\ell t - k_\ell z)} + c.c. \qquad (1.219) \\ &= \frac{1}{2}\mathcal{U}(x,y,z,t)e^{i(\omega_\ell t - k_\ell z)} + c.c. \end{aligned}$$

where $\tilde{\mathcal{E}}$ is the complex pulse envelope and $\tilde{u}$ describes the transverse beam profile. The medium through which the pulse travels is not to be specified. In general, it will respond linearly as well as nonlinearly to the electric field. For example, the pulse changes shape and chirp due to dispersion while it is amplified or absorbed nonlinearly because of a time dependent gain coefficient. Therefore, the wave equation derived before, for the case of linear dispersive media, must be supplemented by certain nonlinear interaction terms. In following chapters we will discuss those nonlinear processes in detail. For the moment we will introduce them only formally. Let us first assume that a change in the beam profile can be neglected. Then the behavior of the field is fully described by its complex envelope $\tilde{\mathcal{E}}$. The propagation equation in local coordinates reads

$$\frac{\partial}{\partial z}\tilde{\mathcal{E}} = \frac{1}{2}ik_\ell''\frac{\partial^2}{\partial t^2}\tilde{\mathcal{E}} - \mathcal{D} + \mathcal{B}_1 + \mathcal{B}_2 + \cdots + \mathcal{B}_n \qquad (1.220)$$

where the terms $\mathcal{B}_i$ stand for contributions from nonlinear light matter interaction. A direct numerical evaluation of Eq. (1.220) often requires solving a set of nonlinear, partial differential equations. Note that for the determination of the $\mathcal{B}_i$, additional (differential) equations describing the medium must be considered. As with partial differential equations in general, the numerical procedures are rather complicated. Moreover, they may differ largely from each other even when the problems seem to be similar from the physical point of view.

A more intuitive approach can be chosen, as outlined next. The sample of length L is divided into M slices of length $\Delta z = L/M$; each slice sufficiently thin as to induce only a small change in the pulse parameters. Assuming that the complex envelope at propagation distance $z = m\Delta z$ $(m = 1, 2, \cdots, M)$ is given by $\tilde{\mathcal{E}}(t, z)$, the envelope at the output of the next slice $(z + \Delta z)$ can be obtained from Eq (1.220) as

$$\begin{aligned}\tilde{\mathcal{E}}(t, z + \Delta z) &= \tilde{\mathcal{E}}(t, z) + \left[\frac{1}{2} i k''_\ell \frac{\partial^2}{\partial t^2} \tilde{\mathcal{E}}(t, z) - \mathcal{D} + \mathcal{B}_1(t, z, \tilde{\mathcal{E}}) \right. \\ &\quad \left. + \mathcal{B}_2(t, z, \tilde{\mathcal{E}}) + \cdots + \mathcal{B}_n(t, z, \tilde{\mathcal{E}}) \right] \Delta z \end{aligned} \tag{1.221}$$

which we can be written formally as

$$\begin{aligned}\tilde{\mathcal{E}}(t, z + \Delta z) &= \tilde{\mathcal{E}}(t, z) + \delta_{k''} \tilde{\mathcal{E}}(t, z) + \delta_{\mathcal{D}} \tilde{\mathcal{E}}(t, z) + \delta_1 \tilde{\mathcal{E}}(t, z) \\ &\quad + \delta_2 \tilde{\mathcal{E}}(t, z) +, \cdots, + \delta_n \tilde{\mathcal{E}}(t, z). \end{aligned} \tag{1.222}$$

The quantities $\delta_i \tilde{\mathcal{E}}(z, t)$ represent the (small) envelope changes due to the various linear and nonlinear processes. For their calculation the envelope at z only is required. The action of the individual processes is treated as if they occur successively and independently in each slice. The pulse envelope at the end of each slice is then the sum of the input pulse plus the different contributions. The resulting envelope $\tilde{\mathcal{E}}(t, z + \Delta z)$ serves as input for the next slice, and so on until $z + \Delta z = L$.

The methods which can be applied to determine $\delta_i \tilde{\mathcal{E}}$ depend on the specific kind of interaction. For example, it may be necessary to solve a set of differential equations, but only with respect to the time coordinate. As mentioned before, the discussion of nonlinear optical processes will be the subject of following chapters.

This type of numerical calculation is critically dependent on the number of slices. It is the strongest interaction affecting the propagating pulse which determines the length of the slices. As a rule of thumb, the envelope distortion in each slice must not exceed a few percent, and doubling and halving of M must not change the results more than the required accuracy allows.

Many propagation problems have been investigated already with ps and ns light pulses, theoretically as well as experimentally. The severe problem when dealing with fs light pulses is dispersion, which enters Eq (1.222) through $\delta_k''\tilde{\mathcal{E}}$ (GVD) and $\delta_{\mathcal{D}}\tilde{\mathcal{E}}$ (higher order dispersion). From the discussion in the preceding sections we can easily derive expressions for these quantities. If only GVD needs to be considered, we can start from Eq (1.110)

$$\tilde{\mathcal{E}}(\Omega, z + \Delta z) = \tilde{\mathcal{E}}(\Omega, z) e^{-ik_\ell''\Omega^2 \Delta z/2} \tag{1.223}$$

which, for sufficiently small Δz, can be approximated as

$$\tilde{\mathcal{E}}(\Omega, z + \Delta z) \approx \tilde{\mathcal{E}}(\Omega, z) - \frac{1}{2} i k_\ell'' \Omega^2 \Delta z \tilde{\mathcal{E}}(\Omega, z). \tag{1.224}$$

Thus we have for $\delta_{k''}\tilde{\mathcal{E}}(t, z)$

$$\delta_{k''}\tilde{\mathcal{E}}(t, z) \approx \mathcal{F}^{-1}\left\{-\frac{1}{2} i k_\ell'' \Omega^2 \Delta z \tilde{\mathcal{E}}(\Omega, z)\right\}. \tag{1.225}$$

If additional dispersion terms matter, we can utilize Eq. (1.174) and obtain

$$\delta_{\mathcal{D}}\tilde{\mathcal{E}}(t, z) = \mathcal{F}^{-1}\left\{-i \sum_{n=3}^{\infty} \frac{1}{n!} k_\ell^{(n)} \Omega^n \Delta z \tilde{\mathcal{E}}(\Omega, z)\right\}. \tag{1.226}$$

Next, let us consider a change in the beam profile. This must be taken into account if the propagation length through the material is long as compared with the confocal length. In addition, beam propagation effects can play a role if the setup to be modelled consists of various individual elements separated from each other by air or vacuum. This is the situation that is, for instance, encountered in lasers. It is the evolution of $\tilde{\mathcal{U}} = \tilde{u}\tilde{\mathcal{E}}$ rather than only that of $\tilde{\mathcal{E}}$ that has to be modelled now. The change of $\tilde{\mathcal{U}}$ from z to $z + \Delta z$ is

$$\tilde{\mathcal{U}}(x, y, z + \Delta z, t) = \tilde{\mathcal{U}}(x, y, z, t) + \delta\tilde{\mathcal{U}} \tag{1.227}$$

where

$$\delta\tilde{\mathcal{U}} = \tilde{u}\delta\tilde{\mathcal{E}} + \tilde{\mathcal{E}}\delta\tilde{u}. \tag{1.228}$$

The change of the pulse envelope $\delta\tilde{\mathcal{E}}$ can be derived as described above. For the determination of $\delta\tilde{u}$ we can evaluate the diffraction integral (1.189) over a propagation length or equivalently proceed to the Fourier space and use Eq. (1.181). For Gaussian beams we may simply use Eq. (1.187).

1.8 Space–time effects

For very short pulses a coupling of spatial and temporal effects becomes important even for propagation in a nondispersive medium. The physical reason is that self–diffraction of a beam of finite transverse size (e.g., Gaussian beam) is wavelength dependent. A separation of time and frequency effects according to Eqs. (1.178) and (1.179) is clearly not feasible if such processes matter. One can construct a solution by solving the diffraction integral (1.189) for each spectral component. The superposition of these solutions and an inverse Fourier–transform then yields the temporal field distribution. Starting with a field $\tilde{E}(x', y', \Omega) = \mathcal{F}\left\{\tilde{E}(x', y', t)\right\}$ in a plane $\Sigma'(x', y')$ at $z = 0$ we find for the field in a plane $\Sigma(x, y)$ at $z = L$:

$$\begin{aligned} \tilde{E}(x, y, L, t) &= \mathcal{F}^{-1}\left\{\frac{i\Omega e^{-i\Omega L/c}}{2\pi c L} \int\int dx'dy'\, \tilde{E}(x', y', \Omega)\right. \\ &\quad \left. \times \exp\left[-i\frac{\Omega}{2Lc}\left((x-x')^2 + (y-y')^2\right)\right]\right\} \end{aligned} \tag{1.229}$$

where we have assumed a nondispersive medium with refractive index $n = 1$. Solutions can be found by solving numerically Eq. (1.229) starting with an arbitrary pulse and beam profile at a plane $z = 0$. Properties of these solutions were discussed by Christov [25]. They revealed that the pulse becomes phase modulated in space and time with a pulse duration that changes across the beam profile. Due to the stronger diffraction of long–wavelength components the spectrum on axis shifts to shorter wavelengths.

For a Gaussian beam and pulse profile at $z = 0$, i.e., $\tilde{E}(x', y', 0, t) \propto \exp(-r'^2/w_0^2)\exp(-t^2/\tau_{G0}^2)\exp(i\omega_\ell t)$ with $r'^2 = x'^2 + y'^2$, the time–space distribution of the field at $z = L$ is of the form [25]:

$$\tilde{E}(r, z = L, t) \propto \exp\left(-\frac{\eta^2}{\tau_G^2}\right)\exp\left[\left(-\frac{w_0\omega_\ell\tau_{G0}}{2Lc\tau_G}r\right)^2\right]\exp\left(i\frac{\omega_\ell\tau_{G0}^2}{\tau_G^2}\eta\right) \tag{1.230}$$

where

$$\tau_G^2 = \tau_{G0}^2 + [w_0 r/(Lc)]^2 \tag{1.231}$$

and $\eta = \left[t - L/c - r^2/(2Lc)\right]$. This result shows a complex mixing of spatial and temporal pulse and beam characteristics. The first term in Eq. (1.230) indicates a pulse duration that increases with increasing distance r from the optical axis. For an order of magnitude estimation let us determine the input pulse duration τ_{G0} at which the pulse duration has increased to $2\tau_{G0}$ at a radial coordinate $r = w$ after the beam has propagated over a certain distance $L >> \rho_0$. From Eq. (1.231) this is equivalent to $\tau_{G0} = w_0 r/(Lc)$. For $r = w$ with $w \approx L\lambda/(\pi w)$, cf. Eq. (1.184), the pulse duration becomes $\tau_{G0} \approx \lambda/(\pi c)$. Obviously, these effects become only important if the pulses approach the single-cycle regime.

1.9 Problems

1. Verify the c_B factors of the pulse–duration–bandwidth–product of a Gaussian and sech-pulse as given in Table 1.1.

2. Calculate the pulse duration $\bar{\tau}_p$ defined as the second moment in Eq. (1.49) for a Gaussian pulse and compare with τ_p (FWHM).

3. Consider a medium consisting of particles that can be described by harmonic oscillators so that the linear susceptibility in the vicinity of a resonance is given by Eq. (1.155). Investigate the behavior of the phase and group velocity in the absorption region. You will find a region where $v_g > v_p$. Is the theory of relativity violated here?

4. Assume a Gaussian pulse which is linearly chirped in a phase modulator that leaves its envelope unchanged. The chirped pulse is then sent through a spectral amplitude–only filter of spectral width (FWHM) $\Delta\omega_F$. Calculate the duration of the filtered pulse and determine an optimum spectral width of the filter for which the shortest pulses are obtained. (Hint: For simplification you may assume an amplitude only filter of Gaussian profile, i.e., $\tilde{H}(\omega - \omega) = \exp\left[-\ln 2\left(\frac{\omega-\omega}{\Delta\omega_F}\right)^2\right]$.)

5. Derive the general expression for $d^n/d\Omega^n$ in terms of derivatives with respect to λ.

6. Assume that both the temporal and spectral envelope functions $\mathcal{E}(t)$ and $\mathcal{E}(\Omega)$, respectively, are peaked at zero. Let us define a pulse duration τ_p^* and spectral width $\Delta\omega_p^*$ using the electric field and its Fourier transform by
$$\tau_p^* = \frac{1}{|\mathcal{E}(t=0)|}\int_{-\infty}^{\infty} |\mathcal{E}(t)|dt$$
and
$$\Delta\omega_p^* = \frac{1}{|\mathcal{E}(\Omega=0)|}\int_{-\infty}^{\infty} |\mathcal{E}(\Omega)|d\Omega.$$

Show that for this particular definition of pulse duration and spectral width the uncertainty relation reads

$$\tau_p^* \Delta\omega_p^* \geq 2\pi.$$

7. Derive Eqs. (1.61) and (1.62). Hint: Make use of Parsival's theorem

$$2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(\Omega)|^2 d\Omega$$

and the fact that $\mathcal{F}\{tf(t)\} = -i\frac{d}{d\Omega}\mathcal{F}\{f(t)\}$.

8. A polarization — to second order in the electric field — is defined as $P^{(2)}(t) \propto \chi^{(2)} E^2(t)$. We have seen that the preferred representation for the field is the complex quantity $E^+(t) = \frac{1}{2}\mathcal{E}(t)\exp[i(\omega t + \varphi(t)]$. Give a convenient description of the nonlinear polarization in terms of $E^+(t)$, $\mathcal{E}(t)$ and $\varphi(t)$. Consider in particular second harmonic generation and optical rectification. Explain the physics associated with the various terms of $P^{(2)}$ (or $P^{+(2)}$, if you can define one).

9. Starting from the one–dimensional wave equation (1.70), show that the slowly–varying envelope approximation corresponds essentially to neglecting self-induced reflection.

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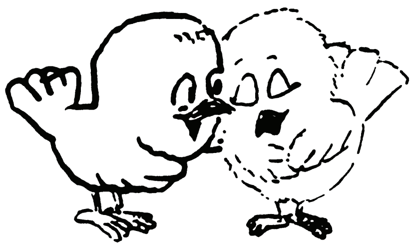
TO CHIRP OR NOT TO CHIRP ...



... THAT IS THE CHALLENGE

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