Electron Plasma Oscillator

Electron gas

Consider an ensemble of charges at equilibrium (electrons in a metal for instance) of density N_e. The electrons and positive charges cannot be in a static equilibrium – there is always some motion unless we are at absolute zero. Above an average electron density N_e, there will be fluctuations of electron density δN. To these fluctuations $\delta N(t)$ – representing a departure from equilibrium, will correspond some local field $E(t)$ and forces on the electron that departed from equilibrium. First task is to derive a second order differential equation for the fluctuation δN, and for the electric field induced by these fluctuations.

In general in a fluid, the velocity is not only a function of time, but also of position. Therefore, the derivative of velocity $\vec{v}$ is:

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{dt} + (\vec{\nabla} \cdot \vec{v})\frac{d\vec{r}}{dt}. = \frac{\partial \vec{v}}{dt} + \vec{v}\vec{\nabla} \cdot \vec{v}). \tag{1}$$

for those familiar with hydrodynamics, this is the Navier Stokes equation. Newton's equation of motion for an electron in such a fluid is

$$m_e \frac{\partial v}{\partial t} + m_e v \nabla v = F \tag{2}$$

F being the force on an electron, with a term proportional to the local field E and the charge, and a term proportional to the magnetic field and the electron velocity.

The conservation equation for the total electrons density N is

$$\frac{\partial N}{\partial t} + \vec{\nabla} \cdot N\vec{v} = 0 \tag{3}$$

One can easily derive this equation by manipulation of Maxwell's equation, but in doing so, one looses completely the physical picture. *Instead, show that the Eq. (3) results from simple logical consideration of the flux of electrons entering an infinitesimal volume.*

To simplify, let us make the following approximations.

1. $N = N_e + \delta N_e$, where δN_e is small compared to n_0.
2. Magnetic field neglected
3. $\vec{v}\nabla\vec{v}$ neglected (correct for an incompressible fluid, or for v small)
4. There is no applied external field, no external source.
5. neglect collisions

No applied field

The conservation equation for the cloud of electrons is:

$$\frac{\partial N_e}{\partial t} + \nabla N v = \frac{\partial N}{\partial t} + \vec{\nabla} \cdot N\vec{v} = \text{Source terms} = 0. \tag{4}$$

The equation of motion is:

$$\frac{\partial v}{\partial t} + v\nabla v = -\frac{e}{m}E_{internal} + \frac{e}{m}E_{applied} \tag{5}$$

where the magnetic field has been neglected – kind of standard approximation when studying local effects, except for the last problem where an external magnetic field is applied. Given the approximations stated above:

$$\frac{\partial \delta N_e}{\partial t} + N_e \nabla v = 0 \tag{6}$$

$$\frac{\partial v}{\partial t} = -\frac{e}{m}E \tag{7}$$

$$\nabla E = -\frac{e}{\epsilon_0}\delta N_e \tag{8}$$

which leads to:

$$\nabla \frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\nabla v = -\frac{1}{N_e}\frac{\partial^2 \delta N_e}{\partial t^2} = -\frac{e}{m}\nabla E = \frac{e^2 \delta N_e}{m\epsilon_0} \tag{9}$$

leading to:

$$\frac{\partial^2 \delta N_e}{\partial t^2} + (\frac{e^2 N_e}{m\epsilon_0})\delta N_e = 0. \tag{10}$$

It is remarkable that both v and E can be eliminated. To get to the field equation, uses the Ampere law:

$$\nabla \times B = \frac{\partial D}{\partial t} - e\delta N_e v = 0 \tag{11}$$

since B is neglected. This leads to:

$$\frac{\partial E}{\partial t} = \frac{e}{\epsilon_0}\delta N_e v \tag{12}$$

and taking the derivative:

$$\frac{\partial^2 E}{\partial t^2} = \frac{e}{\epsilon_0}\frac{\partial (Nv)}{\partial t} \approx \frac{eN_e}{\epsilon_0}\frac{\partial v}{\partial t} = -\left(\frac{e^2 N_e}{m\epsilon_0}\right)E \tag{13}$$

Influence of collisions The collisions enter as a term proportional to the velocity in the equation of motion. As in the classical harmonic oscillator, they will bring a damping term for the fluctuations of density. In presence of collisions, the system of equations for the plasma fluctuations is:

$$\frac{\partial \delta N_e}{\partial t} + N_e \nabla v = 0 \tag{14}$$

$$\frac{\partial v}{\partial t} + \nu_c v = -\frac{e}{m}E \tag{15}$$

$$\nabla E = -\frac{e}{\epsilon_0}\delta n_e \tag{16}$$

which leads to:

$$\nabla\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\nabla v = -\frac{1}{N_e}\frac{\partial^2 \delta N_e}{\partial t^2} = -\frac{e}{m}\nabla E + \frac{\nu_c}{n_0}\frac{\partial \delta N_e}{\partial t} = \frac{e^2 \delta N_e}{m\epsilon_0} \tag{17}$$

leading to:

$$\frac{\partial^2 \delta N_e}{\partial t^2} + \frac{\nu_c}{N_e}\frac{\partial \delta N_e}{\partial t} + (\frac{e^2 N_e 2}{m\epsilon_0})n = 0. \tag{18}$$

Dielectric constant of plasma

An electromagnetic wave of frequency ω is incident on a region containing 10^{16} electrons/cc. The collision rate of the electrons is 10^{10} s^{-1}. (a) Find the dielectric constant of that medium at the frequency ω. (b) Plot the phase velocity of the radiation and the attenuation coefficient in that medium, as a function of frequency.

As for the previous problem, we neglect the term $\vec{v}\vec{\nabla} \cdot \vec{v}$. For the driving field, write $E = \mathcal{E}\exp(i\omega t)$. Choose an axis x along the E field, and z for the direction of propagation. The B field will induce a longitudinal (along z) component of the motion of the electron, which we will neglect. Find the current $N_e e v_x$ due to the applied field, and use it in the right hand side of Ampere law (Biot Savard), to find a dielectric constant.

Equation of motion

$$m_e\frac{dv}{dt} = -eE + e[v \times B] - m_e\nu_c v \tag{19}$$

Electron current For the driving field, we write $E = (1/2)\mathcal{E}\exp(i\omega t)$. Let us choose an axis x along the E field, and z for the direction of propagation. The B field will induce a longitudinal (along z) component of the motion of the electron, which we will neglect. (We will consider a longitudinal applied magnetic field B_0 along z in the last problem of this series.

The main effect of the electromagnetic field will be to impart an electron motion along x, with a velocity v_x, giving rise to a *current* $N_e e v_x$ given by:

$$N_e e v = \frac{N_e e^2 \mathcal{E}}{m_e(\nu_c + i\omega)} \tag{20}$$

Dielectric constant To find the dielectric constant, we use Maxwell's equation:

$$\begin{aligned} \nabla \times H &= i\omega\epsilon_0\mathcal{E} + \frac{N_e e^2 \mathcal{E}}{m_e(\nu_c + i\omega)} && (21) \\ &= i\omega\epsilon_0\mathcal{E}\left[1 - \frac{N_e e^2}{m_e\epsilon_0(\omega^2 + \nu_c^2)} - i\frac{N_e e^2}{m_e\epsilon_0(\omega^2 + \nu_c^2)}\frac{\nu_c}{\omega}\right], && (22) \end{aligned}$$

where we have assumed that there is no magnetic field. One could also attribute the source term on the right of Eqs. (22) to a displacement current:

$$\begin{aligned} \nabla \times H &= \frac{\partial D}{\partial t} = \frac{\partial \epsilon E}{\partial t} && (23) \\ &= i\omega\epsilon\mathcal{E} && (24) \end{aligned}$$

This classical short cut is used to define a complex dielectric constant ϵ:

$$\begin{aligned}\epsilon &= \epsilon_0\left[1-\frac{N_e e^2}{m\epsilon_0(\omega^2+\nu_c^2)}-i\frac{N_e e^2}{m_e\epsilon_0(\omega^2+\nu_c^2)}\frac{\nu_c}{\omega}\right]\\ &= \epsilon_0\left[1-\frac{\omega_p^2}{\omega^2+\nu_c^2}\left(1+i\frac{\nu_c}{\omega}\right)\right]\\ &= \epsilon_r+i\epsilon_i. \end{aligned}\tag{25}$$

The big approximation, questionable in dynamic plasma (in particular when the density is fast increasing with time, is in Eq. (24). Instead, one should write after Eq. (21):

$$\nabla\times H=\frac{\partial\epsilon E}{\partial t}=E\frac{\partial\epsilon}{\partial t}+\epsilon\frac{\partial\epsilon E}{\partial t}=i\omega\epsilon\mathcal{E}+E\frac{\partial\epsilon}{\partial t}\tag{26}$$

CASE I: High frequency $\omega\gg\omega_p$; $\nu_c\ll\omega_p$.

For the given data, $\omega_p\approx 5.645\cdot 10^{12}$ s^{-1}, larger than $\nu_c=10^{10}$ s^{-1}. Let us start at the high frequency end of the scale, where $\omega\gg\omega_p\gg\nu_c$. In the high frequency range, $\epsilon_r\gg\epsilon_i$ [as shown by Eq. (25)], we can make the approximation:

$$\begin{aligned} n &= \sqrt{\frac{\epsilon}{\epsilon_0}}=\sqrt{\frac{\epsilon_r}{\epsilon_0}}\sqrt{1+i\frac{\epsilon_i}{\epsilon_r}}\\ &\approx \sqrt{\frac{\epsilon_r}{\epsilon_0}}\left(1+i\frac{\epsilon_i}{2\epsilon_r}\right)\\ n_r &= \sqrt{1-\frac{\omega_p^2}{\omega^2+\nu_c^2}}\\ n_i &= 0\end{aligned}\tag{27}$$

The phase velocity is therefore approximately given by:

$$\begin{aligned}\frac{c}{n} &\approx c\sqrt{\frac{\epsilon_0}{\epsilon_r}}\\ &= \frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2+\nu_c^2}}}\approx c\left[1+\frac{1}{2}\frac{\omega_p^2}{\omega^2+\nu_c^2}\right].\end{aligned}\tag{28}$$

CASE II: Intermediate frequency ω close to ω_p, with the condition $\nu_c\ll\omega_p$.

One can see from Eq. (28) that the phase velocity will increase as the frequency decreases, to reach a huge value at the frequency ω_c given by $\omega_c^2+\nu_c^2=\omega_p^2$ in Eq. (25). The real part of ϵ goes to zero, and the phase velocity goes to infinity. There is no propagation in the plasma — in fact no penetration, since the reflection coefficient $[(n-1)/n+1)]^2\to 1$ is unity. The plasma has become a perfect mirror.

CASE III: Going to the limit $\omega \to 0$.

As $\omega \to 0$, the real part of ϵ tends towards a constant value:

$$\epsilon_r \to \epsilon_0 \left[1 - \frac{\omega_p^2}{\nu_c^2}\right]. \tag{29}$$

The imaginary part of ϵ tends towards infinity. By substituting in the expression for the index of refraction, one sees that the real and imaginary parts of the index tend to infinity at the same rate as $\sqrt{\epsilon_i}/\sqrt{2\epsilon_0}$.

plots

The plots below use the expression:

$$\tilde{n} = n_r + i \cdot n_i = \sqrt{1 + \chi_r + i.\ \chi_i} \tag{30}$$

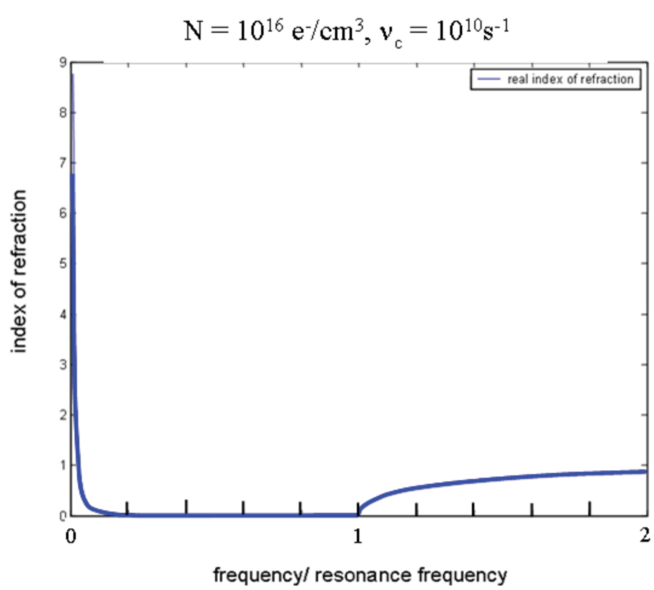

Figure 1: Real part of the index of refraction versus frequency.

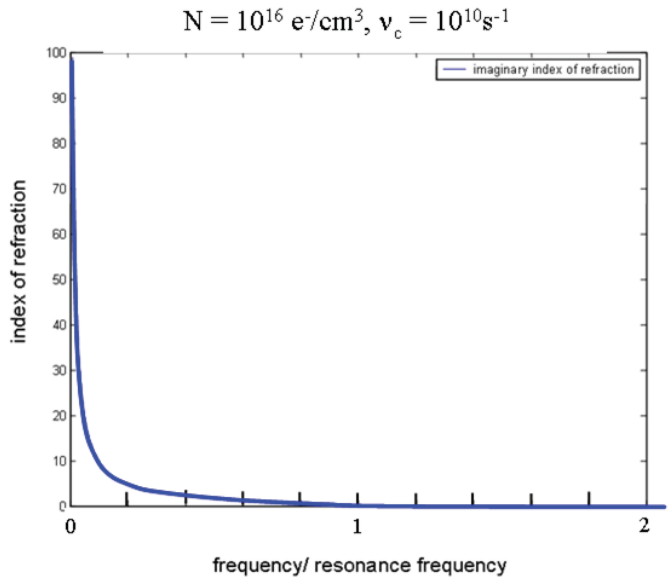

Figure 2: Imaginary part of the index of refraction versus frequency.

Dielectric constant of an ensemble of dipoles

Instead of free electrons, we have now a concentration of 10^{16} *bound* electrons/cc, with a restoring force such that the natural resonance frequency is 10^{15} s^{-1}, and the same collision rate as above. Plot the real and imaginary parts of the index of refraction, as a function of frequency. Find the FWHM of the absorption. Assuming an initial intensity of 10 W/cm^2, and light at the resonance frequency, find an expression for the intensity versus propagation distance in this medium. The dipole moment is $p = qr$, and the polarization of the medium is just the sum of the dipole moment of each atom or molecule. We just need to find r in terms of the other parameters, such as the exciting electric field $E = \mathcal{E}\exp(i\omega t)$. The medium response is assumed to be $r\exp(i\omega t)$. We assume that the local field seen by the electrons is the same as the applied electric field.

The equation of motion for the electrons is now:

$$m\frac{d^2r}{dt^2} = -\omega_0^2 mr - \frac{m}{\tau}\frac{dr}{dt} - eE \tag{31}$$

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$$r = \frac{-e/m}{\omega_0^2 - \omega^2 + i\omega/\tau}. \tag{32}$$

The polarizability of the medium consisting in N non-interacting electron oscillators is:

$$N\alpha = \epsilon_0\chi = \frac{Ne^2/m}{\omega_0^2 - \omega^2 + i\omega/\tau}, \tag{33}$$

from which one can extract the real and imaginary parts:

$$\begin{aligned}\chi_r &= \frac{\omega_p^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}} \\ \chi_i &= \frac{\omega_p^2(\frac{\omega}{\tau})}{(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}}.\end{aligned} \tag{34}$$

Near resonance, we can write:

$$\begin{aligned}n_r = 1 &+ \frac{\omega_p^2(\omega_0-\omega)/4\omega_0}{(\omega_0-\omega)^2+\frac{1}{2\tau^2}} \\ n_i &= \frac{\frac{\omega_p^2}{4\omega_0\tau}}{(\omega_0-\omega)^2+\frac{1}{2\tau^2}}\end{aligned} \tag{35}$$

With the values given, $n_r(\omega = \omega_0) \approx 1$ goes to the maximum value $n_r(\omega = \omega_0 - 1/2\tau) = 1 + 1/8$, and a minimum value of $n_r(\omega = \omega_0 + 1/2\tau) = 1 - 1/8$, where $n_i(\omega = \omega_0) = 1/4$, and the FWHM of the absorption line is $1/\tau$.

Dielectric constant of a medium with magnetic field

Consider the same medium as in the previous problem.

Now a static magnetic field B in the direction of propagation of the electromagnetic wave is added. Show that the left and right circularly polarized electromagnetic waves have different dielectric functions. How does this result influence the propagation of a linearly polarized beam?

Equation of motion Take the z-axis along the direction of propagation. When a static magnetic field $B = Be_z$ is added , the equation of motion for a harmonically bounded particle is

$$m \times \frac{d^2x}{dt^2} = -m\omega_0^2 x + eE + e(\frac{dx}{dt} \times B) \tag{36}$$

As plane electromagnetic was are transverse, E has only x and z components. The component equation are

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -m\omega_0^2 x + eE_x + e(\frac{dy}{dt} \times B_z) \\ m\frac{d^2y}{dt^2} &= -m\omega_0^2 y + eE_y - e(\frac{dx}{dt} \times B_z) \\ m\frac{d^2z}{dt^2} &= -m\omega_0^2 z \end{aligned} \tag{37}$$

Right circularly polarized The last equation shows that motion along the z direction is harmonic but not affected by the applied fields and can thus be neglected. For the right circularly polarized wave

$$E_R = E_0 \cos(\omega t) e_x - E_0 \sin(\omega t) e_y \tag{38}$$

so the remaining equations of motion are

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x + eE_0 \cos(\omega t) + e(\frac{dy}{dt} \times B_z) \tag{39}$$

$$m\frac{d^2y}{dt^2} = -m\omega_0^2 y - eE_0 \sin(\omega t) - e(\frac{dx}{dt} \times B_z) \tag{40}$$

Putting

$$u = x + iy \quad ; \quad \omega_c = \frac{eB}{m} \tag{41}$$

(number) - i(number+1) gives

$$\frac{d^2u}{dt^2} - i \times \omega_c \frac{du}{dt} + \omega_0^2 = \frac{eE_0}{m} e^{i\omega t} \tag{42}$$

In the steady state $u \approx e^{i\omega t}$. substitution in the above gives

$$u = \frac{eE_0(\cos(\omega t) + i\sin(\omega t)}{m(\omega_0^2 - \omega^2 + \omega\omega_c)} \tag{43}$$

Separating the real and imaginary parts we have

$$\begin{aligned} x &= \frac{eE_0(\cos(\omega t)}{m(\omega_0^2 - \omega^2 + \omega\omega_c)} \\ y &= -\frac{eE_0(\sin(\omega t)}{m(\omega_0^2 - \omega^2 + \omega\omega_c)} \end{aligned} \tag{44}$$

combining the above in the vector form gives

$$r = \frac{eE_R}{m(\omega_0^2 - \omega^2 + \omega\omega_c)} \tag{45}$$

Hence the polarization of the medium due to the right circularly polarized wave is

$$P = Ner = \frac{Ne^2E_R}{m(\omega_0^2 - \omega^2 + \omega\omega_c)} \tag{46}$$

As$\epsilon = \epsilon_0(1 + \frac{P}{E})$, the above gives

$$\epsilon_R = \epsilon_0(1 + \frac{Ne^2}{m(\omega_0^2 - \omega^2 + \omega\omega_c)} \tag{47}$$

Left circularly polarized Similarly for the left circularly polarized wave

$$E_L = E_0\cos(\omega t)e_x + E_0\sin(\omega t)e_y \tag{48}$$

we find

$$\epsilon_L = \epsilon_0(1 + \frac{Ne^2}{m(\omega_0^2 - \omega^2 - \omega\omega_c)} \tag{49}$$

The difference between ϵ_R and ϵ_L is therefore

$$\delta\epsilon(\omega) = \epsilon_L - \epsilon_R \;\; = \;\; \epsilon_0\frac{Ne^2}{m}\frac{2eB\omega/(mc)}{(\omega_0^2 - \omega^2)^2 - (eB\omega/mc)^2} \tag{50}$$