Group velocity – there is more to it than a beating of two frequencies

A pulse of light can be seen as a bullet that propagates through a medium, that can be vacuum, a transparent medium, or even an absorbing or amplifying medium. In optics, we are used to decompose a burst of energy into a group of waves. We know that all waves propagate at the speed of light, with some correction for the index of refraction of the medium – thus c/n instead of c. So one would expect the bullet of light to propagate also at that velocity. That this is not the case can be seen through the properties of Fourier transforms.

Propagation through a passive, boring transparent dielectric

The dielectric is characterized by a frequency dependent k vector. One takes the Fourier transform of Maxwell's equation to find that the field should be of the form:

$$E(\Omega, z) = E(\Omega, 0)e^{-ik(\Omega)z} \approx E(\Omega, 0)e^{-i[k_0 + \Omega\frac{dk}{d\Omega}]z}. \tag{1}$$

The Fourier transform of that expression, using the "shift" property:

$$E(t, z) = \mathcal{E}(t - \frac{dk}{d\Omega}z)e^{i\omega t - k_0 z} = \mathcal{E}(t - \frac{z}{v_g})e^{i\omega t - k_0 z} \tag{2}$$

Since $k = 2\pi n(\Omega)/\lambda = \Omega n(\Omega)/c$:

$$\frac{dk}{d\Omega} = \frac{n}{c} - \frac{\omega}{c}\left.\frac{dn}{d\Omega}\right|_{\omega} = \frac{n}{c} + \frac{\lambda_0}{c}\left.\frac{dn}{d\lambda}\right|_{\lambda_0} \tag{3}$$

A bit more general

Now we consider an optical element, like a dielectric mirror, or any structure that changes the phase of the pulse without changing the amplitude. The phase change upon transmission is $\psi(\Omega)$. The transmitted field should be of the form:

$$E_t(\Omega) = E_i(\Omega)e^{i\psi(\Omega)} \approx E_i(\Omega)e^{i[\psi_0 + \Omega\frac{d\psi}{d\Omega}]}. \tag{4}$$

The Fourier transform of that expression, using the "shift" property:

$$E(t, z) = e^{i\psi_0}\mathcal{E}(t - \frac{d\psi}{d\Omega})e^{i\omega t} = \mathcal{E}(t - \tau_d)e^{i\omega t} \tag{5}$$

The linear part of any frequency dependent phase change corresponds to a delay. The case of the phase change proportional to propagation distance (kz) is just a particular case: $\tau_d \rightarrow z/v_g$.

More complicated...

The concepts of "group delay" or "group velocity" make only sense as long as there is a well defined "group" with little or no deformation. This is why the higher orders terms of the Taylor expansion of k were neglected. One can conceive circumstances where the envelope is modified by processes totally independent of the wave, a modification that results in the field envelope being conserved as $\mathcal{E}(t - \frac{z}{v_e})$. In that case, the Fourier transform of the propagating field is:

$$\mathbf{F.T.}\left\{\mathcal{E}(t - \frac{z}{v_e})\right\} e^{i(\omega t - kz)} = \mathcal{E}(\Omega) e^{-i(k_0 + \frac{dk}{d\Omega} + \frac{1}{v_e})z}. \tag{6}$$

One encounters a much more complex situation in transmission of elements that include gain and loss, which modify the pulse shape in a complex fashion. In some cases, it happens that the pulse shaping results in a dynamic equilibrium, a self-sustaining shape. A "soliton" is the most commonly cited example of such a shaping. We have seen that the linearized form of Maxwell's propagation equation reduces to

$$\left[\frac{\partial}{\partial z} + \frac{n}{c}\frac{\partial}{\partial t}\right]\mathcal{E}(t) = \frac{ik}{2\epsilon_0}P(z,t) \tag{7}$$

where P is a complex function of the electric field. That function can be such that a steady-state is established. For instance, the light bullet envelope $\mathcal{E}(t)$ can have saturable gain at the leading edge, and absorption at the trailing edge. An example of such steady state propagation in an absorber is the case where:

$$\frac{\partial \mathcal{E}(t,z)}{\partial z} = -\frac{\alpha}{2}\tanh(t,z)\mathcal{E}(t,x). \tag{8}$$

This equation is written in a *retarded* frame of reference. That means that for each increment of z, the time has been shifted by $t \to t - nz/c$. The steady state solution to that equation is:

$$\mathcal{E}(t,x) = \mathrm{sech}[\frac{t}{\tau_b} - \frac{z}{v_e}], \tag{9}$$

where v_e is the group delay created artificially by the constant pulse shape modification. z/v_e is a group delay to be added to n/c (since we are in the retarded frame of reference). Substituting the bunch shape Eq. (9) into the evolution equation (8) one find a relation between the velocity v_e, the constant α, and the pulse duration τ_b.

There are other examples of solitons which are totally dispersive, encountered in ultrashort pulse lasers and in fibers.

What is sometimes called "Group Velocity" has nothing to do with it...

Here we take as an example the laser gyro, which is a ring laser sensitive to a rotation around an axis perpendicular to its plane. As the laser rotates, the outputs corresponding to the two senses of rotation experience opposite frequency shifts given by:

$$\Delta\nu = \frac{4A}{P\lambda}\Omega, \tag{10}$$

where P is the perimeter of the laser, A the area covered, λ the wavelength, and Ω the angular rotation rate. The condition of resonance has to be satisfied for the two sense of circulation:

$$\begin{aligned} \frac{P_+}{\lambda} &= 2N_+\pi \\ \frac{P_-}{\lambda} &= 2N_-\pi \\ \frac{P_+ - P_-}{\lambda} &= 2q\pi \end{aligned} \tag{11}$$

Taking the difference, one has a resonance condition also, for $N_+ - N_- = q$ One can imagine putting a mirror in the cavity with a very large dispersion at the optical frequency $\Delta\varphi/\Delta\nu$. The resonance condition becomes now:

$$k(P_+ - P_-) + \frac{\Delta\varphi}{\Delta\nu}\Delta\nu = 2q\pi \tag{12}$$

The gyro response is

$$\Delta\nu = \frac{2R}{\lambda}\Omega + \frac{\Delta\varphi}{\Delta\nu}\frac{\Delta\nu}{\tau_{rt}}. \tag{13}$$

The dispersion introduces a feedback such that:

$$\Delta\nu = \frac{\frac{2R\Omega}{\lambda}}{1 - \frac{\Delta\varphi}{\Delta\nu}\frac{1}{\tau_r t}} \tag{14}$$

It should be noted that:

$$\frac{\Delta\varphi}{2\pi\Delta\nu} = \frac{d\psi}{d\Omega} = \tau_d$$

is the definition of a group delay. If this group delay approaches the round-trip time, there is a huge enhancement of gyro response. This is however not a group delay effect.