Maxwell's equations

Contents

1 Derive the wave equation

We begin from first principles i.e. Maxwell's equations in a material:

$$\nabla \cdot D = \rho_f \qquad \text{Gauss' Law} \tag{1}$$

$$\nabla \cdot B = 0 \qquad \text{Gauss' magnetism law} \tag{2}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{Faraday's Law} \tag{3}$$

$$\nabla \times H = \frac{\partial D}{\partial t} + J_f \qquad \text{Ampere's Law} \tag{4}$$

Applying the curl ($\nabla\times$) to both sides of Faraday's law and simplifying (distributive property and curl of curl identity) leads to,

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\nabla \times \frac{\partial B}{\partial t}. \tag{5}$$

Since the curl and time derivative operators commute (as any mixed partial derivative should), they can be interchanged on the right-hand-side (RHS):

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t}\left(\nabla \times B\right). \tag{6}$$

The constitutive relation between the magnetic flux density, B, and the magnetic field strength (or magnetic auxiliary field), H, is,

$$\begin{aligned} B &= \mu_0(H + M) \\ &= \mu_0(H + \chi_m H) \\ &= \mu_0(1 + \chi_m)H \\ &= \mu H. \end{aligned} \tag{7}$$

As an aside, keep in mind that H is the magnetic field in vacuum and B is the total magnetic field. This seems to be opposite of the electric field where E is the field in vacuum and the auxiliary displacement field, D, is the total field. In a non-magnetic material like the one we will consider here, $\mu = \mu_0$. Also, since we are in a dielectric there is no free current, $J_f = 0$. This allows us to plug Ampere's law into Eq. 6:

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \frac{\partial}{\partial t}\frac{\partial D}{\partial t}. \tag{8}$$

The constitutive relations for the displacement and electric field are,

$$\begin{aligned} D &= \epsilon_0 E + P \\ &= \epsilon_0 E + P_L + P_{NL} \\ &= \epsilon_0 E + \epsilon_0 \chi^{(1)} E + P_{NL} \\ &= \epsilon_0 (1 + \chi) E + P_{NL} \\ &= \epsilon E + P_{NL}. \end{aligned} \tag{9}$$

Here we will consider a linear medium so that $P_{NL} = 0$. This means, since $\rho_f = 0$ in the dielectric, that $\nabla \cdot D = \nabla \cdot \epsilon E = 0$. Using the second equality of Eq. 9 results in Eq. 8 taking the form,

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2}. \tag{10}$$

Finally $\mu_0 \epsilon_0 = 1/c^2$ which leads us to the wave equation,

$$\boxed{\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2}.} \tag{11}$$

Other relations that may be useful include,

$$\epsilon_r = \epsilon/\epsilon_0 = 1 + \chi \tag{12}$$

$$\mu_r = \mu/\mu_0 = 1 + \chi_m \tag{13}$$

$$n^2 = \epsilon_r \mu_r \tag{14}$$

2 Derive the slowly-varying wave equation

We start with the 1-Dimensional Ansatz,

$$E = \frac{1}{2}\tilde{\mathcal{E}}(t, z)e^{i(\omega t - kz)}. \tag{15}$$

so that $\nabla \to -\frac{\partial}{\partial z}$ in the wave equation. From Eq. 9 we know that $P_L = \epsilon_0 \chi E$. The wave equation is now,

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\chi}{c^2}\frac{\partial^2 E}{\partial t^2} \tag{16}$$

In an absorbing medium, χ is actually complex. In this case we will specify that n be the index of refraction, or the real part of χ. The constitutive relations can be refined (inserting $\mu_r = 1$ since the dielectric considered here is non-magnetic),

$$\chi = \chi_r + i\chi_i \tag{17}$$

$$\epsilon_r = \epsilon/\epsilon_0 = 1 + \chi_r \tag{18}$$

$$n^2 = \epsilon_r \tag{19}$$

Eq. 16 can be rearranged,

$$\begin{aligned} \frac{\partial^2 E}{\partial z^2} - \frac{1+\chi_r}{c^2}\frac{\partial^2 E}{\partial t^2} &= i\frac{\chi_i}{c^2}\frac{\partial^2 E}{\partial t^2} \\ \frac{\partial^2 E}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} &= i\frac{\chi_i}{c^2}\frac{\partial^2 E}{\partial t^2} \end{aligned} \tag{20}$$

Using the Ansatz of Eq. 15 and the chain rule results in (dropping the explicit z and t amplitude dependence for brevity of notation),[1]

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{2}\frac{\partial^2 \tilde{\mathcal{E}}}{\partial t^2}e^{i(\omega t - kz)} + i\omega\frac{\partial \tilde{\mathcal{E}}}{\partial t}e^{i(\omega t - kz)} - \frac{1}{2}\omega^2\tilde{\mathcal{E}}e^{i(\omega t - kz)} \tag{21}$$

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{2}\frac{\partial^2 \tilde{\mathcal{E}}}{\partial z^2}e^{i(\omega t - kz)} - ik\frac{\partial \tilde{\mathcal{E}}}{\partial z}e^{i(\omega t - kz)} - \frac{1}{2}k^2\tilde{\mathcal{E}}e^{i(\omega t - kz)}. \tag{22}$$

[1] There is a shortcut that can be used by noticing that the LHS of Eq. 16 can be decomposed into left and right propogating waves: $\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \left(\frac{\partial E}{\partial z} - \frac{1}{c}\frac{\partial E}{\partial t}\right)\left(\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t}\right)$. This significantly simplifies the math and leads to the same Eq. 26.

Invoking the slowly varying envelope approximation (SVEO) means that $\partial^2\tilde{\mathcal{E}}/\partial t^2 = \partial^2\tilde{\mathcal{E}}/\partial z^2 = 0$ so Eq. 20 becomes,

$$-ik\frac{\partial\tilde{\mathcal{E}}}{\partial z} - i\frac{n^2\omega}{c^2}\frac{\partial\tilde{\mathcal{E}}}{\partial t} + \left(\frac{n^2\omega^2}{2c^2} - \frac{1}{2}k^2\right)\tilde{\mathcal{E}} = \frac{\omega\chi_i}{c^2}\frac{\partial\tilde{\mathcal{E}}}{\partial t} - i\frac{\omega^2\chi_i}{2c^2}\tilde{\mathcal{E}}. \tag{23}$$

Setting $k = n\omega/c$ causes the first order term on the LHS to become zero, such that we must keep the second order $\partial\mathcal{E}/\partial t$ term. This is not the case on the RHS where the first order term remains and suppresses the effect of the second-order term i.e. $(\omega\chi_i/c^2)(\partial\tilde{\mathcal{E}}/\partial t) \ll (i\omega^2\chi_i/2c^2)\tilde{\mathcal{E}}$. Rearranging leads to,

$$ik\frac{\partial\tilde{\mathcal{E}}}{\partial z} + ik\frac{n}{c}\frac{\partial\tilde{\mathcal{E}}}{\partial t} = -i\frac{k^2\chi_i}{2n^2}\tilde{\mathcal{E}} \tag{24}$$

$$\frac{\partial\tilde{\mathcal{E}}}{\partial z} + \frac{n}{c}\frac{\partial\tilde{\mathcal{E}}}{\partial t} = \frac{k\chi_i}{2n^2}\tilde{\mathcal{E}}. \tag{25}$$

In order to avoid explicitly calculating χ, we will define an effective propogation constant $\beta = k\chi_i/n^2$ in addition to recalling that $n = c/v$.

$$\frac{\partial\tilde{\mathcal{E}}}{\partial z} + \frac{1}{v}\frac{\partial\tilde{\mathcal{E}}}{\partial t} = \frac{\beta}{2}\tilde{\mathcal{E}}. \tag{26}$$

2.1 Retarded frame

First we change coordinates to the retarded frame of reference such that,

$$\begin{aligned} z' &= z \\ t' &= t - \frac{z}{v} \end{aligned} \tag{27}$$

Propogating this through,

$$\begin{aligned} \tilde{\mathcal{E}}(z,t) &\rightarrow \tilde{\mathcal{E}}(z'(z), t'(z,t)) \\ \frac{\partial\tilde{\mathcal{E}}}{\partial z} &= \frac{\partial\tilde{\mathcal{E}}}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial\tilde{\mathcal{E}}}{\partial t'}\frac{\partial t'}{\partial z} \\ &= \frac{\partial\tilde{\mathcal{E}}}{\partial z} - \frac{1}{v}\frac{\partial\tilde{\mathcal{E}}}{\partial t'} \\ \frac{\partial\tilde{\mathcal{E}}}{\partial t} &= \frac{\partial\tilde{\mathcal{E}}}{\partial z'}\frac{\partial z'}{\partial t} + \frac{\partial\tilde{\mathcal{E}}}{\partial t'}\frac{\partial t'}{\partial t} \\ &= \frac{\partial\tilde{\mathcal{E}}}{\partial t'}. \end{aligned} \tag{28}$$

Which means,

$$
\begin{aligned}
\frac{\partial \tilde{\mathcal{E}}}{\partial z} + \frac{1}{v}\frac{\partial \tilde{\mathcal{E}}}{\partial t} &= \frac{\partial \tilde{\mathcal{E}}}{\partial z'} - \frac{1}{v}\frac{\partial \tilde{\mathcal{E}}}{\partial t'} + \frac{1}{v}\frac{\partial \tilde{\mathcal{E}}}{\partial t'} \\
&= \frac{\partial \tilde{\mathcal{E}}}{\partial z'}
\end{aligned}
\tag{29}
$$

Using the change of coordinates of Eq. 29 in Eq. 26 leads to,

$$
\frac{\partial \tilde{\mathcal{E}}}{\partial z'} = \frac{\beta}{2}\tilde{\mathcal{E}}. \tag{30}
$$