

Chapter 1

Fundamentals

1.1 Characteristics of femtosecond light pulses

Femtosecond light pulses are electromagnetic wave packets and as such are fully described by the time and space dependent electric field. In the frame of a semi-classical treatment the propagation of such fields and the interaction with matter are governed by Maxwell's equations with the material response given by a macroscopic polarization. In this first chapter we will summarize the essential notations and definitions used throughout the book. The pulse is characterized by measurable quantities which can be directly related to the electric field. A complex representation of the field amplitude is particularly convenient in dealing with propagation problems of electromagnetic pulses. The next section expands on the choice of field representation.

1.1.1 Complex representation of the electric field

Let us consider first the temporal dependence of the electric field neglecting its spatial and polarization dependence, i.e., $\mathbf{E}(x, y, z, t) = E(t)$. A complete description can be given either in the time or the frequency domain. Even though the measured quantities are real, it is generally more convenient to use complex representation. For this reason, starting with the real $E(t)$, one defines the complex spectrum of the field strength $\tilde{E}(\Omega)$, through the complex Fourier transform (\mathcal{F}):

$$\tilde{E}(\Omega) = \mathcal{F}\{E(t)\} = \int_{-\infty}^{\infty} E(t)e^{-i\Omega t} dt = |\tilde{E}(\Omega)|e^{i\Phi(\Omega)} \quad (1.1)$$

In the definition (1.1), $|\tilde{E}(\Omega)|$ denotes the spectral amplitude and $\Phi(\Omega)$ is the spectral phase. Here and in what follows, complex quantities related to the field are typically written with a tilde.

Since $E(t)$ is a real function, $\tilde{E}(\Omega) = \tilde{E}^*(-\Omega)$ holds. Given $\tilde{E}(\Omega)$, the time dependent electric field is obtained through the inverse Fourier transform (\mathcal{F}^{-1}):

$$E(t) = \mathcal{F}^{-1} \{ \tilde{E}(\Omega) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(\Omega) e^{i\Omega t} d\Omega \quad (1.2)$$

The physical meaning of this Fourier transform is that a pulse can be created by adding a number of waves of different frequency. Figure 1.1 sketches an ultrashort pulse created by adding continuous waves (cw). The waves are shown to be in phase at the time $t = 0$, and add constructively at that point, while destructive interference defines the temporal extension of the pulse. A single isolated pulse in time domain is constructed if the frequency difference between two successive waves is infinitesimal. In the example shown in Fig. 1.1, the frequencies are chosen to be spaced at equal frequency interval $\Delta\omega$, which implies that the same destructive interference takes place at equal time intervals $2\pi/\Delta\omega$. In this picture, the frequency spectrum is composed of a finite number of δ -functions, to which correspond an infinite number of pulses in the time domain.

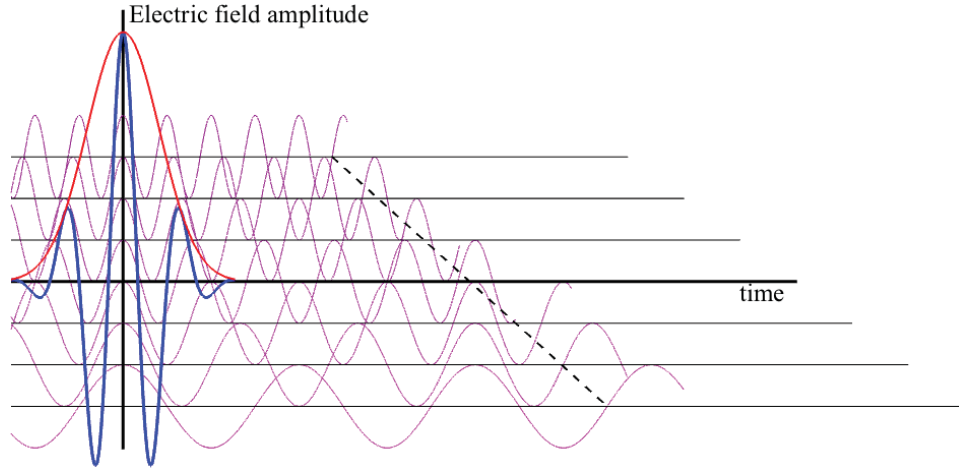


Figure 1.1: Representation of a pulse as a series of cosine waves equally spaced in frequency.

For practical reasons it may not be convenient to use functions which are non-zero for negative frequencies, as needed in the evaluation of Eq. (1.2). Frequently a complex representation of the electric field, also in the time domain, is desired. Both aspects can be satisfied by introducing a complex electric field as

$$\tilde{E}^+(t) = \frac{1}{2\pi} \int_0^{\infty} \tilde{E}(\Omega) e^{i\Omega t} d\Omega \quad (1.3)$$

and a corresponding spectral field strength that contains only positive frequencies:

$$\tilde{E}^+(\Omega) = |\tilde{E}(\Omega)|e^{i\Phi(\Omega)} = \begin{cases} \tilde{E}(\Omega) & \text{for } \Omega \geq 0 \\ 0 & \text{for } \Omega < 0 \end{cases} \quad (1.4)$$

$\tilde{E}^+(t)$ and $\tilde{E}^+(\Omega)$ are related to each other through the complex Fourier transform defined in Eq. (1.1) and Eq. (1.2), i.e.

$$\tilde{E}^+(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}^+(\Omega)e^{i\Omega t} d\Omega \quad (1.5)$$

and

$$\tilde{E}^+(\Omega) = \int_{-\infty}^{\infty} \tilde{E}^+(t)e^{-i\Omega t} dt. \quad (1.6)$$

The real physical electric field $E(t)$ and its complex Fourier transform can be expressed in terms of the quantities derived in Eq. (1.5) and Eq. (1.6) and the corresponding quantities $\tilde{E}^-(t)$, $\tilde{E}^-(\Omega)$ for the negative frequencies. These quantities relate to the real electric field:

$$E(t) = \tilde{E}^+(t) + \tilde{E}^-(t) \quad (1.7)$$

and its complex Fourier transform:

$$\tilde{E}(\Omega) = \tilde{E}^+(\Omega) + \tilde{E}^-(\Omega) \quad (1.8)$$

It can be shown that $\tilde{E}^+(t)$ can also be calculated through analytic continuation of $E(t)$

$$\tilde{E}^+(t) = E(t) + iE'(t) \quad (1.9)$$

where $E'(t)$ and $E(t)$ are Hilbert transforms of each other. In this sense $\tilde{E}^+(t)$ can be considered as the complex analytical correspondent of the real function $E(t)$. The complex electric field $\tilde{E}^+(t)$ is usually represented by a product of an amplitude function and a phase term:

$$\tilde{E}^+(t) = \frac{1}{2}\mathcal{E}(t)e^{i\Gamma(t)} \quad (1.10)$$

In most practical cases of interest here the spectral amplitude will be centered around a mean frequency ω_ℓ and will have appreciable values only in a frequency interval $\Delta\omega$ small compared to ω_ℓ . In the time domain this suggests the convenience of introducing a carrier frequency ω_ℓ and of writing $\tilde{E}^+(t)$ as:

$$\boxed{\tilde{E}^+(t) = \frac{1}{2}\mathcal{E}(t)e^{i\varphi_e}e^{i\varphi(t)}e^{i\omega_\ell t} = \frac{1}{2}\tilde{\mathcal{E}}(t)e^{i\omega_\ell t}} \quad (1.11)$$

where $\varphi(t)$ is the time dependent phase, $\tilde{\mathcal{E}}(t)$ is called the complex field envelope and $\mathcal{E}(t)$ the real field envelope, respectively. The constant phase term $e^{i\varphi_e}$ is most often of no relevance, and can be neglected. There are however particular circumstances pertaining to very short pulses where the outcome of the pulse interaction with matter depends on φ_e , often referred to as “carrier to envelope phase” (CEP). The measurement and control of φ_0 can therefore be quite important. Figure 1.2

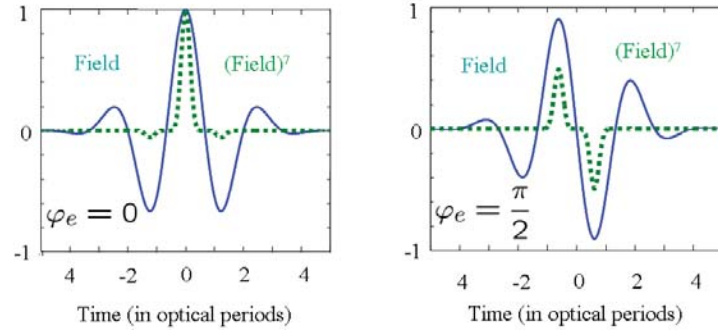


Figure 1.2: Electric field of two extremely short pulses, $E(t) = \exp[-2\ln 2(t/\tau_p)^2]\cos(\omega_\ell t + \varphi_0)$ with $\varphi_0 = 0$ (solid line) and $\varphi_0 = \pi/2$ (dashed line). Both pulses have the same envelope (dotted line). The full width of half maximum of the intensity envelope, τ_p , was chosen as $\tau_p = \pi/\omega_\ell$.

shows the electric field of two pulses with identical $\mathcal{E}(t)$ but different CEP $\varphi_e = 0$ (left) and $\varphi_e = \pi/2$ (right). It is obvious that the difference can be important in the case of highly nonlinear processes, such as for instance a seven’s harmonic generation creating a field proportional to the seventh power of the original field (dotted green lines).

The electric field can formally be represented in a form similar to Eq. (1.11), as illustrated by Fig. 1.2, but the mathematical entity does not always correspond to a physically possible propagating ultrashort pulses. Since the laser pulse represents a propagating electromagnetic wave packet the dc component of its spectrum vanishes. Hence the time integral over the electric field is zero.

$$\int_{-\infty}^{\infty} E(t)dt = \int_{-\infty}^{\infty} E(t)e^{-i(\Omega=0)t} dt = \mathcal{F}\{E(t)\}_{\Omega=0} = 0. \quad (1.12)$$

This not the case of the pulse with null CEP ($\varphi_e = 0$) and even less for its seventh harmonic. The convenience of representing pulse envelopes by a Gaussian or Lorentzian or secant hyperbolic envelope fails to be physical for few cycle pulses. This is illustrated in Fig. 1.3. The Fourier transform of a pulse with real electric field $\mathcal{E}(t)\cos(\omega_\ell t)$ can be constructed by shifting by \pm the carrier frequency the

Fourier transform of the envelope. Since the spectrum of a Gaussian has an infinite extension, the two shifted spectra will overlap at zero frequency, a non physical situation. A pulse of a few optical cycles does exist, but its representation should start with a real spectrum that has no component near zero frequency. We will

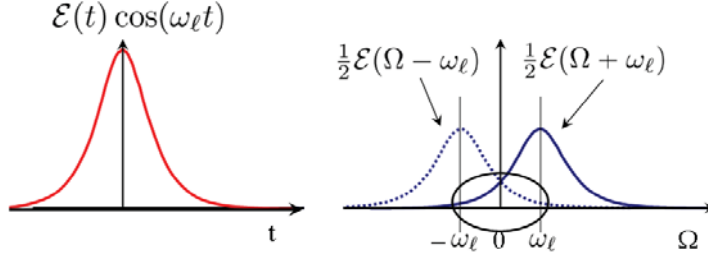


Figure 1.3: A typical pulse representation by, for instance, a Gaussian envelope at a carrier frequency ω_ℓ . The Fourier transform is constructed by shifting the Fourier transform of the envelope by $\pm\omega_\ell$, resulting in un-physical components at and near zero frequency.

discuss the carrier to envelope phase in more detail in Chapters ?? and ??.

While the description of the field given by Eqs. (1.9) through (1.11) is quite general, the usefulness of the concept of an envelope and carrier frequency as defined in Eq. (1.11) is limited to the cases where the bandwidth is only a small fraction of the carrier frequency:

$$\frac{\Delta\omega}{\omega_\ell} \ll 1 \quad (1.13)$$

For inequality (1.13) to be satisfied, the temporal variation of $\mathcal{E}(t)$ and $\varphi(t)$ within an optical cycle $T = 2\pi/\omega_\ell$ ($T \approx 2$ fs for visible radiation) has to be small. The corresponding requirement for the complex envelope $\tilde{\mathcal{E}}(t)$ is

$$\left| \frac{d}{dt} \tilde{\mathcal{E}}(t) \right| \ll \omega_\ell |\tilde{\mathcal{E}}(t)| \quad (1.14)$$

Keeping in mind that today the shortest light pulses contain only a few optical cycles, one has to carefully check whether a slowly varying envelope and phase can describe the pulse behavior satisfactorily. If they do, the theoretical description of pulse propagation and interaction with matter can be greatly simplified by applying the slowly varying envelope approximation (SVEA), as will be evident later in this chapter.

Given the spectral description of a signal, $\tilde{E}^+(\Omega)$, the complex envelope $\tilde{\mathcal{E}}(t)$ is simply the inverse transform of the translated spectral field:

$$\tilde{\mathcal{E}}(t) = \mathcal{E}(t)e^{i\varphi(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\tilde{E}^+(\Omega + \omega_\ell)e^{i\Omega t} d\Omega; \quad (1.15)$$

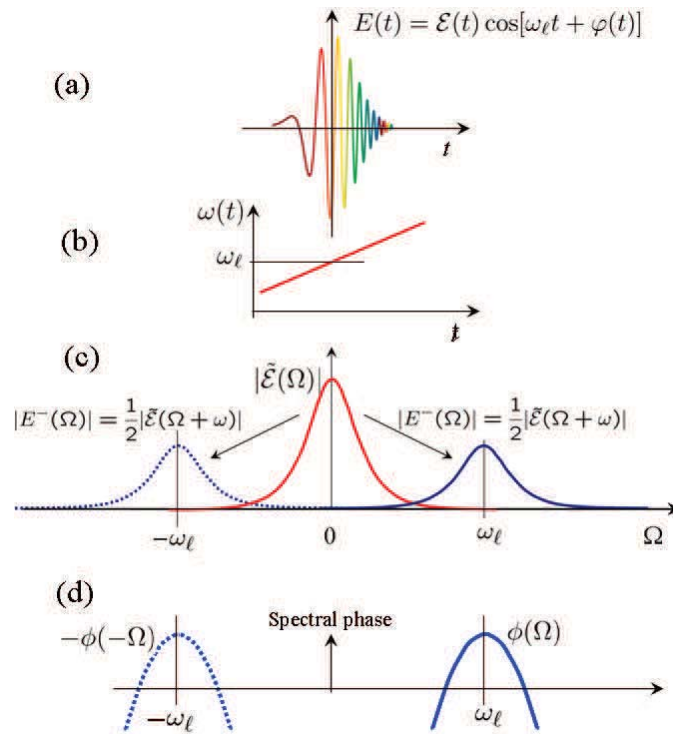


Figure 1.4: (a) Electric field, (b) time dependent carrier frequency, (c) spectral amplitude and (d) spectral phase of a linearly upchirped pulse.

where the modulus $\mathcal{E}(t)$ in Eq. (1.15) represents the real envelope. The optimum “translation” in the spectral domain ω_ℓ is the one that gives the envelope $\tilde{\mathcal{E}}(t)$ with the least amount of modulation. Spectral translation of Fourier transforms is a standard technique to reconstruct the envelope of interference patterns, and is used in Chapter ?? on diagnostic techniques. The Fourier transform of the complex envelope $\tilde{\mathcal{E}}(t)$ is the spectral envelope function:

$$\tilde{\mathcal{E}}(\Omega) = \int_{-\infty}^{\infty} \tilde{\mathcal{E}}(t) e^{-i\Omega t} dt = 2 \int_{-\infty}^{\infty} \tilde{E}^+(t) e^{-i(\Omega + \omega_\ell)t} dt. \quad (1.16)$$

The choice of ω_ℓ is such that the spectral amplitude $\tilde{\mathcal{E}}(\Omega)$ is centered about the origin $\Omega = 0$.

Let us now discuss more carefully the physical meaning of the phase function $\varphi(t)$. The choice of carrier frequency in Eq. (1.11) should be such as to minimize the variation of phase $\varphi(t)$. The first derivative of the phase factor $\Gamma(t)$ in Eq. (1.10) establishes a time dependent carrier frequency (instantaneous frequency):

$$\omega(t) = \omega_\ell + \frac{d}{dt}\varphi(t). \quad (1.17)$$

While Eq. (1.17) can be seen as a straightforward definition of an instantaneous frequency based on the temporal variation of the phase factor $\Gamma(t)$, we will see in Section 1.1.5 that it can be rigorously derived from the Wigner distribution. For $d\varphi/dt = b = \text{const.}$, a non-zero value of b just means a correction of the carrier frequency which is now $\omega'_\ell = \omega_\ell + b$. For $d\varphi/dt = f(t)$, the carrier frequency varies with time and the corresponding pulse is said to be frequency modulated or chirped. For $d^2\varphi/dt^2 < (>)0$, the carrier frequency decreases (increases) along the pulse, which then is called down(up)chirped.

From Eq.(1.10) it is obvious that the decomposition of $\Gamma(t)$ into ω and $\varphi(t)$ is not unique. The most useful decomposition is one that ensures the smallest $d\varphi/dt$ during the intense portion of the pulse. A common practice is to identify ω_ℓ with the carrier frequency at the pulse peak. A better definition — which is consistent in the time and frequency domains — is to use the intensity weighted *average* frequency:

$$\langle \omega \rangle = \frac{\int_{-\infty}^{\infty} |\tilde{\mathcal{E}}(t)|^2 \omega(t) dt}{\int_{-\infty}^{\infty} |\tilde{\mathcal{E}}(t)|^2 dt} = \frac{\int_{-\infty}^{\infty} |\tilde{E}^+(\Omega)|^2 \Omega d\Omega}{\int_{-\infty}^{\infty} |\tilde{E}^+(\Omega)|^2 d\Omega} \quad (1.18)$$

The various notations are illustrated in Fig. 1.4 where a linearly up-chirped pulse is taken as an example. The temporal dependence of the real electric field is sketched in the top part of Fig 1.4. A complex representation in the time domain is illustrated with the amplitude and instantaneous frequency of the field. The

positive and negative frequency components of the Fourier transform are shown in amplitude and phase in the bottom part of the figure.

1.1.2 Power, energy, and related quantities

Let us imagine the practical situation in which the pulse propagates as a beam with cross section A , and with $E(t)$ as the relevant component of the electric field. The (instantaneous) pulse power (in Watt) in a dispersionless material of refractive index n can be derived from the Poynting theorem of electrodynamics [1] and is given by

$$\mathcal{P}(t) = \epsilon_0 cn \int_A dS \frac{1}{T} \int_{t-T/2}^{t+T/2} E^2(t') dt' \quad (1.19)$$

where c is the velocity of light in vacuum, ϵ_0 is the dielectric permittivity and $\int_A dS$ stands for integration over the beam cross section. The power can be measured by a detector (photodiode, photomultiplier etc.) which integrates over the beam cross section. The temporal response of this device must be short as compared to the speed of variations of the field envelope to be measured. The temporal averaging is performed over one optical period $T = 2\pi/\omega_\ell$. Note that the instantaneous power as introduced in Eq. (1.19) is then just a convenient theoretical quantity. In a practical measurement T has to be replaced by the actual response time τ_R of the detector. Therefore, even with the fastest detectors available today ($\tau_R \approx 10^{-13} - 10^{-12}$ s), details of the envelope of fs light pulses can not be resolved directly.

A temporal integration of the power yields the energy \mathcal{W} (in Joules):

$$\mathcal{W} = \int_{-\infty}^{\infty} \mathcal{P}(t') dt' \quad (1.20)$$

where the upper and lower integration limits essentially mean “before” and “after” the pulse under investigation.

The corresponding quantity per unit area is the intensity (W/cm^2):

$$\begin{aligned} I(t) &= \epsilon_0 cn \frac{1}{T} \int_{t-T/2}^{t+T/2} E^2(t') dt' \\ &= \frac{1}{2} \epsilon_0 cn \mathcal{E}^2(t) = 2\epsilon_0 cn \tilde{E}^+(t) \tilde{E}^-(t) = \frac{1}{2} \epsilon_0 cn \tilde{\mathcal{E}}(t) \tilde{\mathcal{E}}^*(t) \end{aligned} \quad (1.21)$$

and the energy density per unit area (J/cm^2):

$$W = \int_{-\infty}^{\infty} I(t') dt' \quad (1.22)$$

Sometimes it is convenient to use quantities which are related to photon numbers, such as the photon flux \mathcal{F} (photons/s) or the photon flux density F (photons/s/cm²):

$$\mathcal{F}(t) = \frac{\mathcal{P}(t)}{\hbar\omega_\ell} \quad \text{and} \quad F(t) = \frac{I(t)}{\hbar\omega_\ell} \quad (1.23)$$

where $\hbar\omega_\ell$ is the energy of one photon at the carrier frequency.

The spectral properties of the light are typically obtained by measuring the intensity of the field, without any time resolution, at the output of a spectrometer. The quantity, called spectral intensity, that is measured is:

$$S(\Omega) = |\eta(\Omega)\tilde{E}^+(\Omega)|^2 \quad (1.24)$$

where η is a scaling factor which accounts for losses, geometrical influences, and the finite resolution of the spectrometer. Assuming an ideal spectrometer, $|\eta|^2$ can be determined from the requirement of energy conservation:

$$|\eta|^2 \int_{-\infty}^{\infty} |\tilde{E}^+(\Omega)|^2 d\Omega = 2\epsilon_0 cn \int_{-\infty}^{\infty} \tilde{E}^+(t)\tilde{E}^-(t)dt \quad (1.25)$$

and Parseval's theorem [2]:

$$\int_{-\infty}^{\infty} |\tilde{E}^+(t)|^2 dt = \frac{1}{2\pi} \int_0^{\infty} |\tilde{E}^+(\Omega)|^2 d\Omega \quad (1.26)$$

from which follows $|\eta|^2 = \epsilon_0 cn/\pi$. The complete expression for the spectral intensity [from Eq. (1.24)] is thus:

$$S(\Omega) = \frac{\epsilon_0 cn}{4\pi} |\tilde{\mathcal{E}}(\Omega - \omega_\ell)|^2. \quad (1.27)$$

Figure 1.5 gives examples of typical pulse shapes and the corresponding spectra.

The complex quantity \tilde{E}^+ will be used most often throughout the book to describe the electric field. Therefore, to simplify notations, we will omit the superscript “+” whenever this will not cause confusion.

1.1.3 Pulse duration and spectral width

Unless specified otherwise, we define the pulse duration τ_p as the full width at half maximum (FWHM) of the intensity profile, $|\tilde{\mathcal{E}}(t)|^2$, and the spectral width $\Delta\omega_p$ as the FWHM of the spectral intensity $|\tilde{\mathcal{E}}(\Omega)|^2$. Making that statement is an obvious admission that other definitions exist. Precisely because of the difficulty of asserting the exact pulse shape, standard waveforms have been selected. The most

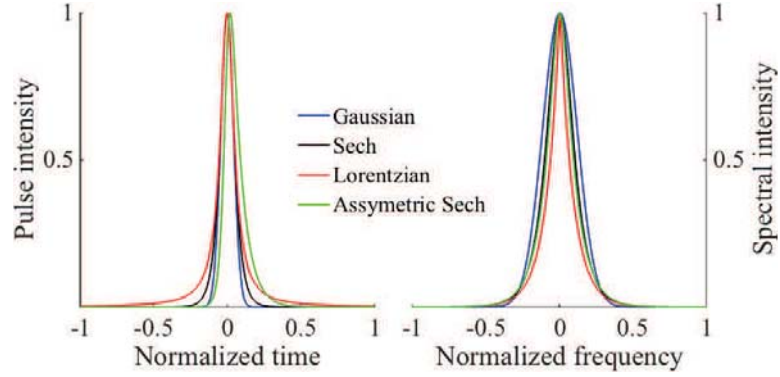


Figure 1.5: Temporal pulse profiles and the corresponding spectra (normalized).

———— Gaussian pulse	$\mathcal{E}(t) \propto \exp[-1.385(t/\tau_p)^2]$
- - - - - sech - pulse	$\mathcal{E}(t) \propto \text{sech}[1.763(t/\tau_p)]$
⋯⋯⋯ Lorentzian pulse	$\mathcal{E}(t) \propto [1 + 1.656(t/\tau_p)^2]^{-1}$
———— asymm. sech pulse	$\mathcal{E}(t) \propto [\exp(t/\tau_p) + \exp(-3t/\tau_p)]^{-1}$

commonly cited are the Gaussian, for which the temporal dependence of the field is:

$$\tilde{\mathcal{E}}(t) = \tilde{\mathcal{E}}_0 \exp\{-(t/\tau_G)^2\} \quad (1.28)$$

and the secant hyperbolic:

$$\tilde{\mathcal{E}}(t) = \tilde{\mathcal{E}}_0 \text{sech}(t/\tau_s). \quad (1.29)$$

The parameters $\tau_G = \tau_p / \sqrt{2 \ln 2}$ and $\tau_s = \tau_p / 1.76$ are generally more convenient to use in theoretical calculations involving pulses with these assumed shapes than the FWHM of the intensity, τ_p .

Since the temporal and spectral characteristics of the field are related to each other through Fourier transforms, the bandwidth $\Delta\omega_p$ and pulse duration τ_p cannot vary independently of each other. There is a minimum duration-bandwidth product:

$$\Delta\omega_p \tau_p = 2\pi\Delta\nu_p \tau_p \geq 2\pi c_B. \quad (1.30)$$

c_B is a numerical constant on the order of 1, depending on the actual pulse shape. Some examples are shown in Table 1.1. The equality holds for pulses without frequency modulation (unchirped) which are called “bandwidth limited” or “Fourier limited”. Such pulses exhibit the shortest possible duration at a given spectral width and pulse shape. We refer the reader to Section 1.1.5, for a more general

Shape	Intensity profile $I(t)$	τ_p FWHM	Spectral profile $S(\Omega)$	$\Delta\omega_p$ FWHM	c_B	$\langle\tau_p\rangle\langle\Delta\Omega_p\rangle$ MSQ
Gauss	$e^{-2(t/\tau_G)^2}$	$1.177\tau_G$	$e^{-\frac{(\Omega\tau_G)^2}{2}}$	$2.355/\tau_G$	0.441	0.5
sech	$\text{sech}^2(t/\tau_s)$	$1.763\tau_s$	$\text{sech}^2\frac{\pi\Omega\tau_s}{2}$	$1.122/\tau_s$	0.315	0.525
Lorentz	$[1 + (t/\tau_L)^2]^{-2}$	$1.287\tau_L$	$e^{-2 \Omega \tau_L}$	$0.693/\tau_L$	0.142	0.7
asym. sech	$[e^{t/\tau_a} + e^{-3t/\tau_a}]^{-2}$	$1.043\tau_a$	$\text{sech}\frac{\pi\Omega\tau_a}{2}$	$1.677/\tau_a$	0.278	
square	1 for $ t/\tau_r \leq 1$, 0 elsewhere	τ_r	$\text{sinc}^2(\Omega\tau_r)$	$2.78/\tau_r$	0.443	3.27

Table 1.1: Examples of standard pulse profiles. The spectral values given are for unmodulated pulses. Note that the Gaussian is the shape with the minimum product of mean square deviation (MSQ) of the intensity and spectral intensity.

discussion of the uncertainty relation between pulse and spectral width based on mean-square deviations.

The shorter the pulse duration, the more difficult it becomes to assert its detailed characteristics. In the femtosecond domain, even the simple concept of pulse duration seems to fade away in a cloud of mushrooming definitions. Part of the problem is that it is difficult to determine the exact pulse shape. For single pulses, the typical representative function that is readily accessible to the experimentalist is the intensity autocorrelation:

$$A_{\text{int}}(\tau) = \int_{-\infty}^{\infty} I(t)I(t-\tau)dt \quad (1.31)$$

The Fourier transform of the correlation (1.31) is the real function:

$$A_{\text{int}}(\Omega) = \tilde{I}(\Omega)\tilde{I}^*(\Omega) \quad (1.32)$$

where the notation $\tilde{I}(\Omega)$ is the Fourier transform of the function $I(t)$, which should not be confused with the spectral intensity $S(\Omega)$. The fact that the autocorrelation function $A_{\text{int}}(\tau)$ is symmetric, hence its Fourier transform is real [2], implies that little information about the pulse shape can be extracted from such a measurement. Furthermore, the intensity autocorrelation (1.31) contains no information about the pulse phase or coherence. This point is discussed in detail in Chapter ??.

1.1.4 Gaussian pulses

Having introduced essential pulse characteristics, it seems convenient to discuss an example to which we can refer to in later chapters. We choose a Gaussian pulse with linear chirp. This choice is one of analytical convenience: the Gaussian shape is *not* the most commonly encountered temporal shape. The electric field is given by

$$\tilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{-(1+ia)(t/\tau_G)^2} \quad (1.33)$$

with the pulse duration

$$\tau_p = \sqrt{2 \ln 2} \tau_G. \quad (1.34)$$

Note that with the definition (1.33) the chirp parameter a is positive for a downchirp ($d\varphi/dt = -2at/\tau_G^2$). The Fourier transform of (1.33) yields

$$\tilde{\mathcal{E}}(\Omega) = \frac{\mathcal{E}_0 \sqrt{\pi} \tau_G}{\sqrt[4]{1+a^2}} \exp \left\{ i\Phi - \frac{\Omega^2 \tau_G^2}{4(1+a^2)} \right\} \quad (1.35)$$

with the spectral phase given by:

$$\phi(\Omega) = -\frac{1}{2} \arctan(a) + \frac{a\tau_G^2}{4(1+a^2)} \Omega^2 \quad (1.36)$$

It can be seen from Eq. (1.35) that the spectral intensity is the Gaussian:

$$S(\omega_\ell + \Omega) = \frac{|\eta|^2 \pi \mathcal{E}_0^2 \tau_G^2}{\sqrt{1+a^2}} \exp \left\{ -\frac{\Omega^2 \tau_G^2}{2(1+a^2)} \right\} \quad (1.37)$$

with a FWHM given by:

$$\Delta\omega_p = 2\pi\Delta\nu_p = \frac{1}{\tau_G} \sqrt{8 \ln 2 (1+a^2)} \quad (1.38)$$

For the pulse duration-bandwidth product we find

$$\Delta\nu_p \tau_p = \frac{2 \ln 2}{\pi} \sqrt{1+a^2} \quad (1.39)$$

Obviously, the occurrence of chirp ($a \neq 0$) results in additional spectral components which enlarge the spectral width and lead to a duration bandwidth product exceeding the Fourier limit ($2 \ln 2 / \pi \approx 0.44$) by a factor $\sqrt{1+a^2}$, consistent with Eq. (1.30). We also want to point out that the spectral phase given by Eq. (1.36) changes quadratically with frequency if the input pulse is linearly chirped. While this is exactly true for Gaussian pulses as can be seen from Eq. (1.36), it holds approximately for other pulse shapes. In the next section, we will develop a concept that allows one to discuss the pulse duration-bandwidth product from a more general point of view and independent of the actual pulse and spectral profile.

1.1.5 Wigner distribution, second order moments, uncertainty relations

Wigner distribution

The Fourier transform as defined in Section 1.1.1 is a widely used tool in beam and pulse propagation. In beam propagation, it leads directly to the far field pattern of a propagating beam (Fraunhofer approximation) of arbitrary transverse profile. Similarly, the Fourier transform leads directly to the pulse temporal profile, following propagation through a dispersive medium, as we will see at the end of this chapter. The Fourier transform gives a weighted average of the spectral components contained in a signal. Unfortunately, the exact spatial or temporal location of these spectral components is hidden in the phase of the spectral field, which is most often not readily available. It is not straightforward to look at the electric field in time and make a statement about the spectral components (and vice versa) without actually taking a Fourier transform. The Wigner function tries to solve this problem by creating a mathematical entity which describes the time and spectral components at the same time, fulfilling the need for new two-dimensional representation of the waves in either the plane of space–wave vector, or time–angular frequency. Such a function was introduced by Wigner [3] and applied to quantum mechanics. The same distribution was applied to the area of signal processing by Ville [4]. Properties and applications of the Wigner distribution in Quantum Mechanics and Optics are reviewed in two recent books by Schleich [5] and Cohen [6]. A clear analysis of the close relationship between Quantum Mechanics and Optics can be found in ref. [7]. In the time–angular-frequency domain, the Wigner distribution of a function $\tilde{E}(t)$ is defined by¹:

$$\begin{aligned} \mathcal{W}_E(t, \Omega) &= \int_{-\infty}^{\infty} \tilde{E}\left(t + \frac{s}{2}\right) \tilde{E}^*\left(t - \frac{s}{2}\right) e^{-i\Omega s} ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}\left(\Omega + \frac{s}{2}\right) \tilde{E}^*\left(\Omega - \frac{s}{2}\right) e^{its} ds \end{aligned} \quad (1.40)$$

One can see that the definition of the Wigner function is a local (i.e. at a given time) representation of the spectrum of the signal, since time integration yields the spectral amplitude:

$$\int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) dt = |\tilde{E}(\Omega)|^2. \quad (1.41)$$

¹ t and Ω are conjugated variables as in Fourier transforms. The same definitions can be made in the space–wavevector domain, where the variables are then x and k .

It is also a local (i.e. at a given spectral component) representation of the signal, since frequency integration yields the temporal intensity:

$$\int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) d\Omega = 2\pi |\tilde{E}(t)|^2 \quad (1.42)$$

In the notation \mathcal{W}_E , the subscript E refers to the use of the instantaneous complex electric field \tilde{E} in the definition of the Wigner function, rather than the electric field envelope $\tilde{\mathcal{E}} = \mathcal{E} \exp[i\omega_\ell t + i\varphi(t)]$ defined at the beginning of this chapter. There is a simple relation between the Wigner distribution \mathcal{W}_E of the instantaneous field \tilde{E} , and the Wigner distribution $\mathcal{W}_\mathcal{E}$ of the real envelope amplitude \mathcal{E} :

$$\begin{aligned} \mathcal{W}_E(t, \Omega) &= \int_{-\infty}^{\infty} \mathcal{E}\left(t + \frac{s}{2}\right) e^{i[\omega_\ell(t+s/2) + \varphi(t+s/2)]} \\ &\quad \times \mathcal{E}^*\left(t - \frac{s}{2}\right) e^{-i[\omega_\ell(t-s/2) + \varphi(t-s/2)]} e^{-i\Omega s} ds \\ &= \int_{-\infty}^{\infty} \mathcal{E}\left(t + \frac{s}{2}\right) \mathcal{E}^*\left(t - \frac{s}{2}\right) e^{-i[\Omega - (\omega_\ell + \dot{\varphi}(t))]s} ds \\ &= \mathcal{W}_\mathcal{E}\{t, [\Omega - (\omega_\ell + \dot{\varphi})]\}. \end{aligned} \quad (1.43)$$

We will drop the subscript “ E ” and “ \mathcal{E} ” for the Wigner function when the distinction is not essential.

The intensity and spectral intensities are directly proportional to frequency and time integrations of the Wigner function. In accordance with Eqs. (1.21) and Eq. (1.27):

$$\frac{1}{2\sqrt{\mu_0/\epsilon}} \int_{-\infty}^{\infty} \mathcal{W}_\mathcal{E}(t, \Omega) d\Omega = I(t) \quad (1.44)$$

$$\frac{1}{2\sqrt{\mu_0/\epsilon}} \int_{-\infty}^{\infty} \mathcal{W}_\mathcal{E}(t, \Omega) dt = S(\Omega). \quad (1.45)$$

Figure 1.6 shows the Wigner distribution of an unchirped Gaussian pulse ((a), left) versus a Gaussian pulse with a linear chirp (quadratic phase modulation) ((b), right). The introduction of a quadratic phase modulation leads to a tilt (rotation) and flattening of the distribution. This distortion of the Wigner function results directly from the relation (1.43) applied to a Gaussian pulse. We have defined in Eq. (1.33) the phase of the linearly chirped pulse as $\varphi(t) = -at^2/\tau_G^2$. If $\mathcal{W}_{\text{unchirp}}$ is the Wigner distribution of the unchirped pulse, the linear chirp transforms that function into:

$$\mathcal{W}_{\text{chirp}} = \mathcal{W}_{\text{unchirp}}\left(t, \Omega - \frac{2at}{\tau_G^2}\right), \quad (1.46)$$

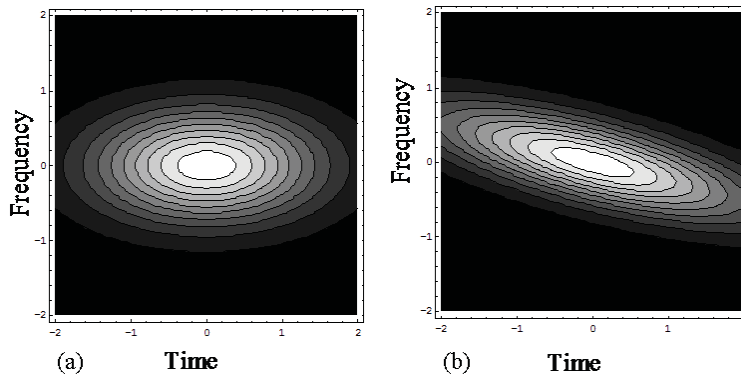


Figure 1.6: Wigner distribution for a Gaussian pulse. Left (a), the phase function $\varphi(t) = \varphi_0$ is a constant. On the right (b), Wigner distribution for a linearly chirped pulse, i.e. with a quadratic phase modulation $\varphi(t) = \alpha t^2$. The elliptical curves are lines of equal Wigner function intensity. The intensity is graded from 0 (black) to the peak (white).

hence the tilt observed in Fig. 1.6. Mathematical tools have been developed to produce a pure rotation of the phase space (t, Ω) . We refer the interested reader to the literature for details on the Wigner distribution and in particular on the fractional Fourier transform [8, 9]. It has been shown that such a rotation describes the propagation of a pulse through a medium with a quadratic dispersion (index of refraction being a quadratic function of frequency) [10].

Moments of the electric field

It is mainly history and convenience that led to the adoption of the FWHM of the pulse intensity as the quantity representative of the pulse duration. Sometimes pulse duration and spectral width defined by the FWHM values are not suitable measures. This is, for instance, the case in pulses with substructure or broad wings causing a considerable part of the energy to lie outside the range given by the FWHM. In these cases it may be preferable to use averaged values derived from the appropriate second-order moments. It appears in fact, as will be shown in examples of propagation, that the second moment of the field distribution is a better choice.

For the sake of generality, let us designate by $f(x)$ the field as a function of the variable x (which can be the transverse coordinate, transverse wave vector, time or frequency). The moment of order n for the quantity x with respect to intensity is

defined as:

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} x^n |f(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \quad (1.47)$$

The first order moment, $\langle x \rangle$, is the “center of mass” of the intensity distribution, and is most often chosen as reference, in such a way as to have a zero value. For example, the center of the transverse distribution will be on axis, $x = 0$, or a Gaussian temporal intensity distribution $\mathcal{E}_0 \exp[-(t/\tau_G)^2]$ will be centered at $t = 0$. A good criterium for the width of a distribution is the mean square deviation (MSQ):

$$\langle \Delta x \rangle = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}. \quad (1.48)$$

The explicit expressions in the time and frequency domains are:

$$\langle \tau_p \rangle = \langle \Delta t \rangle = \left[\frac{1}{W} \int_{-\infty}^{\infty} t^2 I(t) dt - \frac{1}{W^2} \left(\int_{-\infty}^{\infty} t I(t) dt \right)^2 \right]^{\frac{1}{2}} \quad (1.49)$$

$$\langle \Delta \omega_p \rangle = \langle \Delta \Omega \rangle = \left[\frac{1}{W} \int_{-\infty}^{\infty} \Omega^2 S(\Omega) d\Omega - \frac{1}{W^2} \left(\int_{-\infty}^{\infty} \Omega S(\Omega) d\Omega \right)^2 \right]^{\frac{1}{2}} \quad (1.50)$$

where $S(\Omega)$ is the spectral intensity defined in Eq. (1.24). Whenever appropriate we will assume that the first-order moments are zero, which yields $\langle \Delta x \rangle = \sqrt{\langle x^2 \rangle}$.

The second moments can also be defined using the Wigner distribution [Eq. (1.40)]:

$$\langle t^2 \rangle = \frac{\int \int_{-\infty}^{\infty} t^2 \mathcal{W}_E(t, \Omega) dt d\Omega}{\int \int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) dt d\Omega} = \frac{\int_{-\infty}^{\infty} t^2 |\tilde{E}(t)|^2 dt}{\int_{-\infty}^{\infty} |\tilde{E}(t)|^2 dt} \quad (1.51)$$

$$\langle \Omega^2 \rangle = \frac{\int \int_{-\infty}^{\infty} \Omega^2 \mathcal{W}_E(t, \Omega) dt d\Omega}{\int \int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) dt d\Omega} = \frac{\int_{-\infty}^{\infty} \Omega^2 |\tilde{E}(\Omega)|^2 d\Omega}{\int_{-\infty}^{\infty} |\tilde{E}(\Omega)|^2 d\Omega} \quad (1.52)$$

While the above equations do not bring anything new, the Wigner distribution lets us define another quantity, which describes the coupling between conjugated variables:

$$\langle t, \Omega \rangle = \frac{\int \int_{-\infty}^{\infty} (t - \langle t \rangle)(\Omega - \langle \Omega \rangle) \mathcal{W}_E(t, \Omega) dt d\Omega}{\int \int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) dt d\Omega}. \quad (1.53)$$

A non-zero $\langle t, \Omega \rangle$ implies that the center of mass of the spectral intensity evolves with time, as in Fig. 1.6. One can thus define an instantaneous frequency:

$$\omega(t) = \frac{\int_{-\infty}^{\infty} \Omega \mathcal{W}_E(t, \Omega) d\Omega}{\int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) d\Omega}. \quad (1.54)$$

By substituting the definition of the Wigner distribution Eq. (1.40) in Eq. (1.54), it is possible to demonstrate rigourously the relation (1.17). Indeed, substituting the definition (1.43) in Eq. (1.54) leads to:

$$\begin{aligned}
 \omega(t) &= \frac{\int_{-\infty}^{\infty} \Omega \mathcal{W}_E[t, \Omega - (\omega_\ell + \dot{\varphi})] d\Omega}{\int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) d\Omega} \\
 &= \frac{\int_{-\infty}^{\infty} [\Omega' + \omega_\ell + \dot{\varphi}(t)] \mathcal{W}_E[t, \Omega'] d\Omega'}{\int_{-\infty}^{\infty} \mathcal{W}_E(t, \Omega) d\Omega} \\
 &= \omega_\ell + \dot{\varphi}(t),
 \end{aligned} \tag{1.55}$$

where we used the fact that $\int \Omega' \mathcal{W}_E(t, \Omega') d\Omega' = 0$.

There is a well known uncertainty principle between the second moment of conjugated variables. If k is the Fourier-conjugated variable of x , it is shown in Appendix A that:

$$\langle x^2 \rangle \langle k^2 \rangle = \frac{M^4}{4} \geq \frac{1}{4}, \tag{1.56}$$

where we have defined a shape factor “ M^2 ”, which has been extensively used to describe the departure of beam profile from the “ideal Gaussian” [11]. This relation can be applied to time and frequency:

$$\langle t^2 \rangle \langle \Omega^2 \rangle = \frac{M^4}{4} \geq \frac{1}{4}. \tag{1.57}$$

Equality only holds for a Gaussian pulse (beam) shape free of any phase modulation, which implies that the Wigner distribution for a Gaussian shape occupies the smallest area in the time/frequency plane. It is also important to note that the uncertainty relations (1.56) and (1.57) only hold for the pulse widths defined as the mean square deviation. For a Gaussian pulses defined by its electric field $\mathcal{E}(t) = \mathcal{E}_0 \exp[-(t/\tau_G)^2]$:

$$\begin{aligned}
 \langle t^2 \rangle &= \frac{\tau_G^2}{4} \\
 \langle \Omega^2 \rangle &= \frac{1}{\tau_G^2}.
 \end{aligned} \tag{1.58}$$

The product of the two numbers is indeed 1/4, the minimum of the inequality (1.57). while for the products of the full width at half maximum (FWHM) of the intensity and spectral intensity (generally referred to as the “time-bandwidth product” $c_B = \tau_p \Delta \nu_p = 0.441$). In fact, the pulse time-bandwidth product product *is not minimum* for a Gaussian pulse, as illustrated in Table 1.1, which gives the value of

c_B for various pulse shapes without phase modulation. It remains that, for a given pulse shape, c_B is the smallest for pulses without frequency modulation (unchirped) which are called “bandwidth limited” or “Fourier limited”. Such pulses exhibit the shortest possible duration at a given spectral width and pulse shape.

If there is a frequency variation across a pulse, its spectrum will contain additional spectral components. Consequently, the modulated pulse possesses a spectral width which is larger than the Fourier limit given by column five in Table 1.1.

Relation to Quantum Mechanics

The Heisenberg uncertainty relation is contained directly in Eqs (1.56) and (1.57), when taking into account particle wave duality. Indeed, a moving particle with energy $W = p^2/2m$ has an associated wave packet centered at the frequency $\omega = W/\hbar$. This is where the Planck constant enters into the uncertainty relation (1.57). The wave packet has a frequency distribution of second moment $\langle \Omega^2 \rangle$, related by inverse Fourier transform to the temporal distribution, with a second moment in time $\langle t^2 \rangle$, leading to the relation:

$$\langle t^2 \rangle \langle W^2 \rangle = \frac{M^4}{4} \geq \frac{\hbar^2}{4}. \quad (1.59)$$

In space, the wave packet representing the particle has a momentum $k = p/\hbar$. Hence, Equation (1.56) applied to the wave representation of a particle is the Heisenberg uncertainty relation in space:

$$\langle x^2 \rangle \langle k^2 \rangle = \frac{M^4}{4} \geq \frac{\hbar^2}{4}, \quad (1.60)$$

Chirped pulses

A quadratic phase modulation plays an essential role in light propagation, be it in time or space. Since a spherical wavefront can be approximated by a quadratic phase ($\varphi(x) \propto x^2$, where x is the transverse dimension) near any propagation axis of interest, imparting a quadratic spatial phase modulation will lead to focusing or de-focusing of a beam. The analogue is true in time: imparting a quadratic phase modulation ($\varphi(t) \propto t^2$) will lead to pulse compression or broadening after propagation through a dispersive medium. These problems relating to pulse propagation will be discussed in several sections and chapters of this book. In this section we attempt to clarify quantitatively the relation between a quadratic chirp in the temporal or frequency space, and the corresponding broadening of the spectrum or pulse duration, respectively. The results are interchangeable from frequency to temporal space.

Let us first assume that a laser pulse, initially unchirped, propagates through a dispersive material that leaves the pulse spectrum, $|\tilde{\mathcal{E}}(\Omega)|^2$, unchanged but produces a quadratic phase modulation in the frequency domain. The pulse spectrum is centered at the average frequency $\langle\Omega\rangle = \omega_\ell$. The average frequency does not change, hence the first nonzero term in the Taylor expansion of $\phi(\Omega)$ is

$$\phi(\Omega) = \frac{1}{2} \left. \frac{d^2\phi}{d\Omega^2} \right|_0 \langle\Omega^2\rangle, \quad (1.61)$$

where $\phi(\Omega)$ determines the phase factor of $\tilde{\mathcal{E}}(\Omega)$:

$$\tilde{\mathcal{E}}(\Omega) = \mathcal{E}(\Omega)e^{i\phi(\Omega)}. \quad (1.62)$$

The first and second order moments are, according to the definitions (1.47):

$$\langle t \rangle = \frac{\int_{-\infty}^{\infty} t \tilde{\mathcal{E}}(t) \tilde{\mathcal{E}}(t)^* dt}{\int_{-\infty}^{\infty} |\tilde{\mathcal{E}}(t)|^2 dt} = \frac{\int_{-\infty}^{\infty} \frac{d\tilde{\mathcal{E}}(\Omega)}{d\Omega} \tilde{\mathcal{E}}^*(\Omega) d\Omega}{\int_{-\infty}^{\infty} |\tilde{\mathcal{E}}(\Omega)|^2 d\Omega} = \left\langle \frac{d\phi}{d\Omega} \right\rangle \quad (1.63)$$

and

$$\begin{aligned} \langle t^2 \rangle &= \frac{\int_{-\infty}^{\infty} t \tilde{\mathcal{E}}(t) t \tilde{\mathcal{E}}(t)^* dt}{\int_{-\infty}^{\infty} |\tilde{\mathcal{E}}(t)|^2 dt} = \frac{\int_{-\infty}^{\infty} \left| \frac{d\tilde{\mathcal{E}}(\Omega)}{d\Omega} \right|^2 d\Omega}{\int_{-\infty}^{\infty} |\tilde{\mathcal{E}}(\Omega)|^2 d\Omega} \\ &= \frac{\int_{-\infty}^{\infty} \left[\frac{d\mathcal{E}(\Omega)}{d\Omega} \right]^2 d\Omega}{\int_{-\infty}^{\infty} |\mathcal{E}(\Omega)|^2 d\Omega} + \left\langle \left(\frac{d\phi}{d\Omega} \right)^2 \right\rangle. \end{aligned} \quad (1.64)$$

It is left to a problem at the end of this chapter to derive these results. Since the initial pulse was unchirped and its spectral amplitude is not affected by propagation through the transparent medium, the first term in Eq. (1.64) represents the initial second order moment $\langle t^2 \rangle_0$. Substituting the expression for the quadratic phase Eq. (1.61) into Eq. (1.47) for the first order moment, we find from Eq. (1.64):

$$\langle t^2 \rangle = \langle t^2 \rangle_0 + \left[\left. \frac{d^2\phi}{d\Omega^2} \right|_0 \right]^2 \langle\Omega^2\rangle. \quad (1.65)$$

The frequency chirp introduces a temporal broadening (of the second order moment) directly proportional to the square of the chirp coefficient, $\left[\left. \frac{d^2\phi}{d\Omega^2} \right|_0 \right]^2$.

Likewise we can analyze the situation where a temporal phase modulation $\varphi(t) = \left. \frac{d\varphi}{dt} \right|_0 t^2$ is impressed upon the pulse while the pulse envelope, $|\tilde{\mathcal{E}}(t)|^2$, remains unchanged. This temporal frequency modulation or chirp, characterized by

the second derivative in the middle (center of mass) of the pulse, leads to a spectral broadening given by:

$$\langle \Omega^2 \rangle = \langle \Omega^2 \rangle_0 + \left[\left. \frac{d^2 \varphi}{dt^2} \right|_0 \right]^2 \langle t^2 \rangle \quad (1.66)$$

where $\langle \Omega^2 \rangle_0$ refers to the spectrum of the input pulse and $\langle t^2 \rangle$ is the (constant) second-order moment of time.

Equations (1.65) and (1.66) demonstrate the advantage of using the mean square deviation to define the pulse duration and bandwidth, since it shows a simple relation between the broadening in the time or spectral domain, due to a chirp in the spectral or time domain, respectively independent of the pulse and spectral shape. For the two different situations described by Eqs. (1.65) and (1.66), we can apply the uncertainty relation, Eq. (1.57),

$$\langle t^2 \rangle \langle \Omega^2 \rangle = \frac{M^4}{4} \kappa_c \geq \frac{1}{4}. \quad (1.67)$$

We have introduced a factor of chirp κ_c , equal to

$$\kappa_c = 1 + \frac{M^4}{4 \langle t^2 \rangle_0^2} \left[\left. \frac{d^2 \phi}{d\Omega^2} \right|_0 \right]^2 \quad (1.68)$$

in case of a frequency chirp and constant spectrum, or

$$\kappa_c = 1 + \frac{M^4}{4 \langle \Omega^2 \rangle_0^2} \left[\left. \frac{d^2 \varphi}{dt^2} \right|_0 \right]^2 \quad (1.69)$$

in case of a temporal chirp and constant pulse envelope.

In summary, using the *mean square deviation* to define the pulse duration and bandwidth:

- the duration—bandwidth product $\sqrt{\langle t^2 \rangle \langle \Omega^2 \rangle}$ is minimum (0.5) for a Gaussian pulse shape, without phase modulation.
- For any pulse shape, one can define a shape factor M^2 equal to the minimum duration—bandwidth product for that particular shape.
- Any quadratic phase modulation — or linear chirp — whether in frequency or time, increases the bandwidth duration product by a chirp factor κ_c . The latter increases proportionally to the second derivative of the phase modulation, whether in time or in frequency.

1.2 Pulse propagation

So far we have considered only temporal and spectral characteristics of light pulses. In this subsection we shall be interested in the propagation of such pulses through matter. This is the situation one always encounters when working with electromagnetic wave packets (at least until somebody succeeds in building a suitable trap). The electric field, now considered in its temporal and spatial dependence, is again a suitable quantity for the description of the propagating wave packet. In view of the optical materials that will be investigated, we can neglect external charges and currents and confine ourselves to nonmagnetic permeabilities and uniform media. A wave equation can be derived for the electric field vector \mathbf{E} from Maxwell equations (see for instance Ref. [12]) which in Cartesian coordinates reads

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{E}(x, y, z, t) = \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{P}(x, y, z, t), \quad (1.70)$$

where μ_0 is the magnetic permeability of free space. The source term of Eq. (1.70) contains the polarization \mathbf{P} and describes the influence of the medium on the field as well as the response of the medium. Usually the polarization is decomposed into two parts:

$$\mathbf{P} = \mathbf{P}^L + \mathbf{P}^{NL}. \quad (1.71)$$

The decomposition of Eq. (1.71) is intended to distinguish a polarization that varies linearly (\mathbf{P}^L) from one that varies nonlinearly (\mathbf{P}^{NL}) with the field. Historically, \mathbf{P}^L represents the medium response in the frame of “ordinary” optics, e.g., classical optics [13], and is responsible for effects such as diffraction, dispersion, refraction, linear losses and linear gain. Frequently, these processes can be attributed to the action of a host material which in turn may contain sources of a nonlinear polarization \mathbf{P}^{NL} . The latter is responsible for nonlinear optics [14, 15, 16] which includes, for instance, saturable absorption and gain, harmonic generation and Raman processes.

As will be seen in Chapters 3 and ??, both \mathbf{P}^L and in particular \mathbf{P}^{NL} are often related to the electric field by complicated differential equations. One reason is that no physical phenomenon can be truly instantaneous. In this chapter we will omit \mathbf{P}^{NL} . Depending on the actual problem under consideration, \mathbf{P}^{NL} will have to be specified and added to the wave equation as a source term.

1.2.1 The reduced wave equation

Equation (1.70) is of rather complicated structure and in general can solely be solved by numerical methods. However, by means of suitable approximations and

simplifications, one can derive a “reduced wave equation” which will enable us to deal with many practical pulse propagation problems in a rather simple way. We assume the electric field to be linearly polarized and propagating in the z -direction as a plane wave, i.e., the field is uniform in the transverse x, y direction. The wave equation has now been simplified to:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E(z, t) = \mu_0 \frac{\partial^2}{\partial t^2} P^L(z, t) \quad (1.72)$$

As known from classical electrodynamics [12] the linear polarization of a medium is related to the field through the dielectric susceptibility χ . In the frequency domain we have

$$\tilde{P}^L(\Omega, z) = \epsilon_0 \chi(\Omega) \tilde{E}(\Omega, z) \quad (1.73)$$

which is equivalent to a convolution integral in the time domain

$$P^L(t, z) = \epsilon_0 \int_{-\infty}^t dt' \chi(t') E(z, t-t'). \quad (1.74)$$

Here ϵ_0 is the permittivity of free space. The finite upper integration limit, t , expresses the fact that the response of the medium must be causal. For a non-dispersive medium (which implies an “infinite bandwidth” for the susceptibility, $\chi(\Omega) = \text{const.}$) the medium response is instantaneous, i.e., memory free. In general, $\chi(t)$ describes a finite response time of the medium which, in the frequency domain, means nonzero dispersion. This simple fact has important implications for the propagation of short pulses and time varying radiation in general. We will refer to this point several times in later chapters — in particular when dealing with coherent interaction.

The Fourier transform of (1.72) together with (1.73) yields

$$\boxed{\left[\frac{\partial^2}{\partial z^2} + \Omega^2 \epsilon(\Omega) \mu_0 \right] \tilde{E}(z, \Omega) = 0} \quad (1.75)$$

where we have introduced the dielectric constant

$$\epsilon(\Omega) = [1 + \chi(\Omega)] \epsilon_0. \quad (1.76)$$

For now we will assume a real susceptibility and dielectric constant. Later we will discuss effects associated with complex quantities. The general solution of (1.75) for the propagation in the $+z$ direction is

$$\tilde{E}(\Omega, z) = \tilde{E}(\Omega, 0) e^{-ik(\Omega)z}, \quad (1.77)$$

where the propagation constant $k(\Omega)$ is determined by the dispersion relation of linear optics

$$k^2(\Omega) = \Omega^2 \epsilon(\Omega) \mu_0 = \frac{\Omega^2}{c^2} n^2(\Omega), \quad (1.78)$$

and $n(\Omega)$ is the refractive index of the material. For further consideration we expand $k(\Omega)$ about the carrier frequency ω_ℓ

$$k(\Omega) = k(\omega_\ell) + \delta k, \quad (1.79)$$

where

$$\delta k = \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} (\Omega - \omega_\ell) + \frac{1}{2} \left. \frac{d^2k}{d\Omega^2} \right|_{\omega_\ell} (\Omega - \omega_\ell)^2 + \dots \quad (1.80)$$

and write Eq. (1.77) as

$$\tilde{E}(\Omega, z) = \tilde{E}(\Omega, 0) e^{-ik_\ell z} e^{-i\delta k z}, \quad (1.81)$$

where $k_\ell^2 = \omega_\ell^2 \epsilon(\omega_\ell) \mu_0 = \omega_\ell^2 n^2(\omega_\ell) / c^2$. In most practical cases of interest, the Fourier amplitude will be centered around a mean wave vector k_ℓ , and will have appreciable values only in an interval Δk small compared to k_ℓ . In analogy to the introduction of an envelope function slowly varying in time, after the separation of a rapidly oscillating term, cf. Eqs. (1.11)–(1.14), we can define now an amplitude which is slowly varying in the spatial coordinate

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{E}(\Omega + \omega_\ell, 0) e^{-i\delta k z}. \quad (1.82)$$

Again, for this concept to be useful we must require that

$$\left| \frac{d}{dz} \tilde{\mathcal{E}}(\Omega, z) \right| \ll k_\ell |\tilde{\mathcal{E}}(\Omega, z)| \quad (1.83)$$

which implies a sufficiently small wave number spectrum

$$\left| \frac{\Delta k}{k_\ell} \right| \ll 1. \quad (1.84)$$

In other words, the pulse envelope must not change significantly while travelling through a distance comparable with the wavelength $\lambda_\ell = 2\pi/\omega_\ell$. Fourier transforming of Eq. (1.81) into the time domain gives

$$\tilde{E}(t, z) = \frac{1}{2} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} d\Omega \tilde{E}(\Omega, 0) e^{-i\delta k z} e^{i(\Omega - \omega_\ell)t} \right\} e^{i(\omega_\ell t - k_\ell z)} \quad (1.85)$$

which can be written as

$$\boxed{\tilde{E}(t, z) = \frac{1}{2} \tilde{\mathcal{E}}(t, z) e^{i(\omega_\ell t - k_\ell z)}} \quad (1.86)$$

where $\tilde{\mathcal{E}}(t, z)$ is now the envelope varying slowly in space and time, defined by the term in the curled brackets in Eq. (1.85).

Further simplification of the wave equation requires a corresponding equation for $\tilde{\mathcal{E}}$ utilizing the envelope properties. Only a few terms in the expansion of $k(\Omega)$ and $\epsilon(\Omega)$, respectively, will be considered. To this effect we expand $\epsilon(\Omega)$ as series around ω_ℓ , leading to the following form for the linear polarization (1.73)

$$\tilde{P}^L(\Omega, z) = \left(\epsilon(\omega_\ell) - \epsilon_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n \epsilon}{d\Omega^n} \Big|_{\omega_\ell} (\Omega - \omega_\ell)^n \right) \tilde{\mathcal{E}}(\Omega, z). \quad (1.87)$$

In terms of the pulse envelope, the above expression corresponds in the time domain to

$$\begin{aligned} \tilde{P}^L(t, z) &= \frac{1}{2} \left\{ [\epsilon(\omega_\ell) - \epsilon_0] \tilde{\mathcal{E}}(t, z) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} (-i)^n \frac{\epsilon^{(n)}(\omega_\ell)}{n!} \frac{\partial^n}{\partial t^n} \tilde{\mathcal{E}}(t, z) \right\} e^{i(\omega_\ell t - k_\ell z)}, \end{aligned} \quad (1.88)$$

where $\epsilon^{(n)}(\omega_\ell) = \frac{\partial^n}{\partial \Omega^n} \epsilon \Big|_{\omega_\ell}$. The term in the curled brackets defines the slowly varying envelope of the polarization, $\tilde{\mathcal{P}}^L$. The next step is to replace the electric field and the polarization in the wave equation (1.72) by Eq. (1.85) and Eq. (1.88), respectively. We transfer thereafter to a coordinate system (η, ξ) moving with the group velocity $v_g = \left(\frac{dk}{d\Omega} \Big|_{\omega_\ell} \right)^{-1}$, which is the standard transformation to a “retarded” frame of reference:

$$\xi = z \quad \eta = t - \frac{z}{v_g} \quad (1.89)$$

and

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi} - \frac{1}{v_g} \frac{\partial}{\partial \eta}; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}. \quad (1.90)$$

A straightforward calculation leads to the final result:

$$\frac{\partial}{\partial \xi} \tilde{\mathcal{E}} - \frac{i}{2} k_\ell'' \frac{\partial^2}{\partial \eta^2} \tilde{\mathcal{E}} + \mathcal{D} = -\frac{i}{2k_\ell} \frac{\partial}{\partial \xi} \left(\frac{\partial}{\partial \xi} - \frac{2}{v_g} \frac{\partial}{\partial \eta} \right) \tilde{\mathcal{E}} \quad (1.91)$$

The quantity

$$\begin{aligned} \mathcal{D} &= -\frac{i\mu_0}{2k_\ell} \sum_{n=3}^{\infty} \frac{(-i)^n}{n!} \left[\omega_\ell^2 \epsilon^{(n)}(\omega_\ell) - 2n\omega_\ell \epsilon^{(n-1)}(\omega_\ell) \right. \\ &\quad \left. + n(n-1) \epsilon^{(n-2)}(\omega_\ell) \right] \frac{\partial^n}{\partial \eta^n} \tilde{\mathcal{E}} \end{aligned} \quad (1.92)$$

contains dispersion terms of higher order, and has been derived by taking directly the second order derivative of the polarization defined by the product of envelope and fast oscillating terms in Eq. (1.88). The indices of the three resulting terms have been re-defined to factor out a single derivative of order (n) of the field envelope. The second derivative of k :

$$\begin{aligned} k''_\ell &= \left. \frac{\partial^2 k}{\partial \Omega^2} \right|_{\omega_\ell} = - \left. \frac{1}{v_g^2} \frac{dv_g}{d\Omega} \right|_{\omega_\ell} \\ &= \frac{1}{2k_\ell} \left[\frac{2}{v_g^2} - 2\mu_0\epsilon(\omega_\ell) - 4\omega_\ell\mu_0\epsilon^{(1)}(\omega_\ell) - \omega_\ell^2\mu_0\epsilon^{(2)}(\omega_\ell) \right] \end{aligned} \quad (1.93)$$

is the group velocity dispersion (GVD) parameter. It should be mentioned that the GVD is usually defined as the derivative of v_g with respect to λ , $dv_g/d\lambda$, related to k'' through

$$\frac{dv_g}{d\lambda} = \frac{\Omega^2 v_g^2}{2\pi c} \frac{d^2 k}{d\Omega^2}. \quad (1.94)$$

So far we have not made any approximations and the structure of Eq. (1.91) is still rather complex. However, we can exploit at this point the envelope properties (1.14) and (1.83), which, in this particular situation, imply:

$$\left| \frac{1}{k_\ell} \left(\frac{\partial}{\partial \xi} - \frac{2}{v_g} \frac{\partial}{\partial \eta} \right) \tilde{\mathcal{E}} \right| = \left| \frac{1}{k_\ell} \left(\frac{\partial}{\partial z} - \frac{1}{v_g} \frac{\partial}{\partial t} \right) \tilde{\mathcal{E}} \right| \ll |\tilde{\mathcal{E}}| \quad (1.95)$$

The right-hand side of (1.91) can thus be neglected if the prerequisites for introducing pulse envelopes are fulfilled. This procedure is called slowly varying envelope approximation (SVEA) and reduces the wave equation to first-order derivatives with respect to the spatial coordinate.

If the propagation of very short pulses is computed over long distances, the cumulative error introduced by neglecting the right hand side of Eq. (1.91) may be significant. In those cases, a direct numerical treatment of the second order wave equation is required.

Further simplifications are possible for a very broad class of problems of practical interest, where the dielectric constant changes slowly over frequencies within the pulse spectrum. In those cases, terms with $n \geq 3$ can be omitted too ($\mathcal{D} = 0$), leading to a greatly simplified reduced wave equation:

$$\boxed{\frac{\partial}{\partial \xi} \tilde{\mathcal{E}}(\eta, \xi) - \frac{i}{2} k''_\ell \frac{\partial^2}{\partial \eta^2} \tilde{\mathcal{E}}(\eta, \xi) = 0} \quad (1.96)$$

which describes the evolution of the complex pulse envelope as it propagates through a loss-free medium with GVD. The reader will recognize the structure of the one-dimensional Schrödinger equation.

1.2.2 Retarded frame of reference

In the case of zero GVD [$k''_\ell = 0$ in Eq. (1.96)], the pulse envelope does not change at all in the system of local coordinates (η, ξ) . This illustrates the usefulness of introducing a coordinate system moving at the group velocity. In the laboratory frame, the pulse travels at the group velocity without any distortion.

In dealing with short pulses as well as in dealing with white light (see Chapter 2) the appropriate “retarded frame of reference” is moving at the *group* rather than at the *wave* (phase) velocity. Indeed, while a monochromatic wave of frequency Ω travels at the phase velocity $v_p(\Omega) = c/n(\Omega)$, it is the superposition of many such waves with differing phase velocities that leads to a wave packet (pulse) propagating with the group velocity. The importance of the frame of reference moving at the group velocity is such that, in the following chapters, the notation z and t will be substituted for ξ and η , unless the laboratory frame is explicitly specified.

Some propagation problems — such as the propagation of coupled waves in nonlinear crystals discussed in Chapter 3 — are more appropriately treated in the frequency domain. As a simple exercise, let us derive the group velocity directly from the solution of the wave equation in the form of Eq. (1.81)

$$\tilde{E}(\Omega, z) = \tilde{E}(\Omega, 0)e^{-ik_\ell z}e^{-i\delta k z}. \quad (1.97)$$

The Fourier transform amplitude $E(\Omega, 0)$ represented on the top left of Fig. 1.7 is not changed by propagation. On the top right, the time domain representation of the pulse, or the inverse transform of $E(\Omega, 0)$, is centered at $t = 0$ (solid line). We assume that the expansion of the wave vector $k(\Omega)$, Eq. (1.79), can be terminated after the linear term, that is

$$\delta k = \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} (\Omega - \omega_\ell) \quad (1.98)$$

The inverse Fourier-transform of Eq. (1.97) now yields

$$\begin{aligned} \tilde{E}(t, z) &= e^{-ik_\ell z} \int_{-\infty}^{\infty} \tilde{E}(\Omega, 0) \exp \left[-i \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} (\Omega - \omega_\ell) z \right] e^{i\Omega t} d\Omega \\ &= e^{i(\omega_\ell t - k_\ell z)} \int_{-\infty}^{\infty} \tilde{E}(\Omega' + \omega_\ell, 0) \exp \left[i \left(t - \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} z \right) \Omega' \right] d\Omega' \end{aligned} \quad (1.99)$$

where we substituted $\Omega = \Omega' + \omega_\ell$ to obtain the last equation. This equation is just the inverse Fourier-transform of the field spectrum shifted to the origin (i.e., the spectrum of the envelope $\tilde{\mathcal{E}}(\Omega)$, represented on the lower left of Fig. 1.7) with the Fourier variable “time” now given by $t - \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} z$. Carrying out the transform yields

$$\tilde{E}(t, z) = \frac{1}{2} \tilde{\mathcal{E}}(t, z) e^{i(\omega_\ell t - k_\ell z)} = \frac{1}{2} \tilde{\mathcal{E}} \left(t - \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} z, 0 \right) e^{i(\omega_\ell t - k_\ell z)}. \quad (1.100)$$

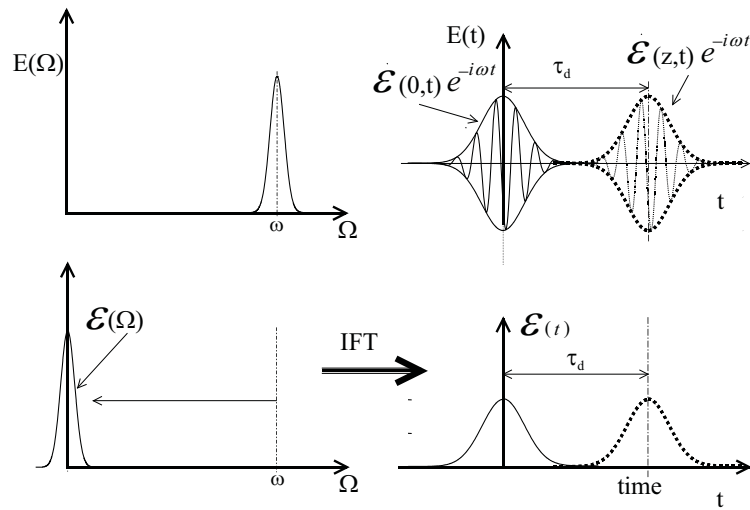


Figure 1.7: The Fourier transform amplitude ($E(\Omega, 0)$) is sketched in the upper left, and the corresponding field in the time domain on the upper right (solid line). The lower part of the figure displays the field amplitudes, $\mathcal{E}(\Omega)$ on the left, centered at the origin of the frequency scale, and the corresponding inverse Fourier transform $\mathcal{E}(t)$. Propagation in the frequency domain is obtained by multiplying the field at $z = 0$ by the phase factor $\exp(-i\tau_d\Omega)$, where $\tau_d = z/v_g$ is the group delay. In the time domain, this corresponds to delaying the pulse by an amount τ_d (right). The delayed fields $|E(z, t)|$ and $\mathcal{E}(z, t)$ are shown in dotted lines on the right of the figure.

We have thus the important result that, in the time domain, the light pulse has been delayed by an amount ($\tau_d = \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} z$) proportional to distance. Within the approximation that the wave vector is a linear function of frequency, the pulse is seen to propagate without distortion with a constant group velocity v_g given by either of the three expressions:

$$\frac{1}{v_g} = \left. \frac{dk}{d\Omega} \right|_{\omega_\ell} \quad (1.101)$$

$$\frac{1}{v_g} = \frac{n_0}{c} + \frac{\omega_\ell}{c} \left. \frac{dn}{d\Omega} \right|_{\omega_\ell} \quad (1.102)$$

$$\frac{1}{v_g} = \frac{n_0}{c} - \frac{\lambda}{c} \left. \frac{dn}{d\lambda} \right|_{\lambda}. \quad (1.103)$$

The first term in Eqs. (1.102) and (1.103) represent the phase delay per unit length, while the second term in these equations is the change in carrier to envelope phase per unit length. We note that the dispersion of the wave vector ($dk/d\Omega$) or of the index of refraction ($dn/d\lambda$) is responsible for a difference between the phase velocity $v_p = c/n_0$ and the group velocity v_g . In a frame of reference moving at the velocity v_g , $\tilde{\mathcal{E}}(z, t)$ remains identically unchanged. Pulse distortions thus only result from high order (higher than 1) terms in the Taylor series expansion of $k(\Omega)$. For this reason, most pulse propagation problems are treated in a retarded frame of reference, moving at the velocity v_g .

Forward/Backward propagating waves

We consider an ultrashort pulse plane wave propagating through a dielectric medium. Before the arrival of the pulse, there are no induced dipoles, and for the index of refraction we assume that of a vacuum ($n = 1$). As the dipoles are driven into motion by the first few cycles of the pulse, the index of refraction changes to the value n of the dielectric. One consequence of this causal phenomenon is the ‘‘precursor’’ predicted by Sommerfeld and Brillouin, see for example [12]. One might wonder if the discontinuity in index created by a short and intense pulse should not lead to a reflection for a portion of the pulse? This is an important question regarding the validity of the first order approximation to Maxwell’s propagation equations. If, at $t = 0$, a short wave packet is launched in the $+z$ direction in a homogeneous medium, is it legitimate to assume that there will be no pulse generated in the opposite direction?

The answer that we give in this section is that, in the framework of Maxwell’s second order equation and a linear polarization, there is no such ‘‘induced reflection’’. This property extends even to the nonlinear polarization created by the interaction of the light with a two-level system.

If we include the non-resonant part of the linear polarization in the index of refraction n (imaginary part of n), the remainder polarization P including all non-linear and resonant interaction effects, adding a phenomenological scattering term σ leads to the following form of the second order wave equation:

$$\left(\frac{\partial^2}{\partial z^2} \tilde{E} - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right) \tilde{E} = \mu_0 \frac{\partial^2}{\partial t^2} \tilde{P} + \frac{n\sigma}{c} \frac{\partial}{\partial t} \tilde{E} \quad (1.104)$$

The polarization appearing in the right hand side can be instantaneous, or be the solution of a differential equation as in the case of most interactions with resonant atomic or molecular systems. Resonant light-matter interactions will be studied in detail in Chapters 3 and ???. The wave equation Eq. (1.104) can be written as a product of a forward and backward propagating operator. Instead of the variables t and z , it is more convenient to use the retarded time variable corresponding to the two possible wave velocities $\pm c/n$:

$$\begin{aligned} s &= t - \frac{n}{c}z \\ r &= t + \frac{n}{c}z. \end{aligned} \quad (1.105)$$

In the new variables, Maxwell's equation (1.104) becomes:

$$\frac{\partial^2}{\partial s \partial r} \tilde{E} = \frac{c^2}{n^2} \left\{ \frac{\mu_0}{4} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right)^2 \tilde{P} + \frac{n\sigma}{c} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right) \right\} \tilde{E}. \quad (1.106)$$

We seek a solution in the form of a forward and a backward propagating field of amplitude $\tilde{\mathcal{E}}_F$ and $\tilde{\mathcal{E}}_B$:

$$\tilde{E} = \frac{1}{2} \tilde{\mathcal{E}}_F e^{i\omega_\ell s} + \frac{1}{2} \tilde{\mathcal{E}}_B e^{i\omega_\ell r}. \quad (1.107)$$

Substitution into Maxwell's Eq. (1.104):

$$\begin{aligned} & e^{i\omega_\ell s} \left[2i\omega_\ell \frac{\partial}{\partial r} + \frac{\partial^2}{\partial s \partial r} + \frac{c\sigma}{2n} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} + 2i\omega_\ell \right) \right] \frac{1}{2} \tilde{\mathcal{E}}_F \\ + & e^{i\omega_\ell r} \left[2i\omega_\ell \frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial r} + \frac{c\sigma}{2n} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} + 2i\omega_\ell \right) \right] \frac{1}{2} \tilde{\mathcal{E}}_B \\ = & -\frac{\mu_0 c^2}{4n^2} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right)^2 \tilde{P}, \end{aligned} \quad (1.108)$$

which we re-write in an abbreviated way using the differential operators \mathcal{L} and \mathcal{M} for the forward and backward propagating waves, respectively:

$$\mathcal{L} \tilde{\mathcal{E}}_F e^{i\omega_\ell s} + \mathcal{M} \tilde{\mathcal{E}}_B e^{i\omega_\ell r} = -\frac{\mu_0 c^2}{4n^2} \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r} \right)^2 \tilde{P}. \quad (1.109)$$

In the case of a linear medium, the forward and backward wave travel independently. If, as initial condition, we choose $\tilde{\mathcal{E}}_B = 0$ along the line $r + s = 0$ ($t = 0$), there will be no back scattered wave. If the polarization is written as a slowly varying amplitude:

$$\tilde{\mathcal{P}} = \frac{1}{2}\tilde{\mathcal{P}}_F e^{i\omega_\ell s} + \frac{1}{2}\tilde{\mathcal{P}}_B e^{i\omega_\ell r}, \quad (1.110)$$

the equations for the forward and backward propagating wave also separate if $\tilde{\mathcal{P}}_F$ is only a function of $\tilde{\mathcal{E}}_F$, and $\tilde{\mathcal{P}}_B$ only a function of $\tilde{\mathcal{E}}_B$. This is because a source term for $\tilde{\mathcal{P}}_B$ can only be formed by a ‘‘grating’’ term, which involves a product of $\tilde{\mathcal{E}}_B \tilde{\mathcal{E}}_F$. It applies to a polarization created by near resonant interaction with a two-level system, using the semi-classical approximation, as will be considered in Chapters 3 and ???. The separation between forward and backward travelling waves has been demonstrated by Eilbeck [17, 18] outside of the slowly-varying approximation. Within the slowly varying approximation, we generally write that the second derivative with respect to time of the polarization as $-\omega_\ell^2 \tilde{\mathcal{P}}$, and therefore, the forward and backward propagating waves are still uncoupled, even when $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(\tilde{\mathcal{E}}_F, \tilde{\mathcal{E}}_B)$, provided there is only a forward propagating beam as initial condition.

1.2.3 Dispersion

For nonzero GVD ($k_\ell'' \neq 0$) the propagation problem (1.96) can be solved either directly in the time or in the frequency domain. In the first case, the solution is given by a Poisson-integral [19] which here reads

$$\tilde{\mathcal{E}}(t, z) = \frac{1}{\sqrt{2\pi i k_\ell'' z}} \int_{-\infty}^t \tilde{\mathcal{E}}(t', z=0) \exp\left(i \frac{(t-t')^2}{2k_\ell'' z}\right) dt' \quad (1.111)$$

As we will see in subsequent chapters, it is generally more convenient to treat linear pulse propagation through transparent linear media in the frequency domain, since only the phase factor of the envelope $\tilde{\mathcal{E}}(\Omega)$ is affected by propagation.

It follows directly from the solution of Maxwell’s equations in the frequency domain [for instance Eqs. (1.77) and (1.82)] that the spectral envelope after propagation through a thickness z of a linear transparent material is given by:

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{\mathcal{E}}(\Omega, 0) \exp\left(-\frac{i}{2} k_\ell'' \Omega^2 z - \frac{i}{3!} k_\ell''' \Omega^3 z - \dots\right). \quad (1.112)$$

Thus we have for the temporal envelope

$$\tilde{\mathcal{E}}(t, z) = \mathcal{F}^{-1} \left\{ \tilde{\mathcal{E}}(\Omega, 0) \exp\left(-\frac{i}{2} k_\ell'' \Omega^2 z - \frac{i}{3!} k_\ell''' \Omega^3 z - \dots\right) \right\}. \quad (1.113)$$

If we limit the Taylor expansion of k to the GVD term k''_ℓ , we find that an initially bandwidth-limited pulse develops a spectral phase with a quadratic frequency dependence, resulting in chirp.

We had defined a “chirp coefficient”

$$\kappa_c = 1 + \frac{M^4}{4\langle t^2 \rangle_0^2} \left[\left. \frac{d\phi}{d\Omega} \right|_{\omega_\ell} \right]^2$$

when considering in Section 1.1.5 the influence of quadratic chirp on the uncertainty relation Eq. (1.67) based on the successive moments of the field distribution. In the present case, we can identify the phase modulation:

$$\left. \frac{d\phi}{d\Omega} \right|_{\omega_\ell} = -k''_\ell z \quad (1.114)$$

Since the spectrum (in amplitude) of the pulse $|\tilde{\mathcal{E}}(\Omega, z)|^2$ remains constant [as shown for instance in Eq. (1.112)], the spectral components responsible for chirp must appear at the expense of the envelope shape, which has to become broader.

At this point we want to introduce some useful relations for the characterization of the dispersion. The dependence of a dispersive parameter can be given as a function of either the frequency Ω or the vacuum wavelength λ . The first, second and third order derivatives are related to each other by

$$\frac{d}{d\Omega} = -\frac{\lambda^2}{2\pi c} \frac{d}{d\lambda} \quad (1.115)$$

$$\frac{d^2}{d\Omega^2} = \frac{\lambda^2}{(2\pi c)^2} \left(\lambda^2 \frac{d^2}{d\lambda^2} + 2\lambda \frac{d}{d\lambda} \right) \quad (1.116)$$

$$\frac{d^3}{d\Omega^3} = -\frac{\lambda^3}{(2\pi c)^3} \left(\lambda^3 \frac{d^3}{d\lambda^3} + 6\lambda^2 \frac{d^2}{d\lambda^2} + 6\lambda \frac{d}{d\lambda} \right) \quad (1.117)$$

The dispersion of the material is described by either the frequency dependence $n(\Omega)$ or the wavelength dependence $n(\lambda)$ of the index of refraction. The derivatives of the propagation constant used most often in pulse propagation problems, expressed in terms of the index n , are:

$$\frac{dk}{d\Omega} = \frac{n}{c} + \frac{\Omega}{c} \frac{dn}{d\Omega} = \frac{1}{c} \left(n - \lambda \frac{dn}{d\lambda} \right) \quad (1.118)$$

$$\frac{d^2k}{d\Omega^2} = \frac{2}{c} \frac{dn}{d\Omega} + \frac{\Omega}{c} \frac{d^2n}{d\Omega^2} = \left(\frac{\lambda}{2\pi c} \right) \frac{1}{c} \left(\lambda^2 \frac{d^2n}{d\lambda^2} \right) \quad (1.119)$$

$$\frac{d^3k}{d\Omega^3} = \frac{3}{c} \frac{d^2n}{d\Omega^2} + \frac{\Omega}{c} \frac{d^3n}{d\Omega^3} = -\left(\frac{\lambda}{2\pi c} \right)^2 \frac{1}{c} \left(3\lambda^2 \frac{d^2n}{d\lambda^2} + \lambda^3 \frac{d^3n}{d\lambda^3} \right) \quad (1.120)$$

The second equation, Eq. (1.119), defining the group velocity dispersion (GVD) is the frequency derivative of $1/v_g$. Multiplied by the propagation length L , it describes the frequency dependence of the group delay. It is sometimes expressed in $\text{fs}^2 \mu\text{m}^{-1}$.

A positive GVD corresponds to

$$\frac{d^2k}{d\Omega^2} > 0 \quad (1.121)$$

1.2.4 Gaussian pulse propagation

For a more quantitative picture of the influence that GVD has on the pulse propagation we consider the linearly chirped Gaussian pulse of Eq. (1.33)

$$\tilde{\mathcal{E}}(t, z=0) = \mathcal{E}_0 e^{-(1+ia)(t/\tau_{G0})^2} = \mathcal{E}_0 e^{-(t/\tau_{G0})^2} e^{i\varphi(t, z=0)}$$

entering the sample. To find the pulse at an arbitrary position z , we multiply the field spectrum, Eq. (1.35), with the propagator $\exp(-i\frac{1}{2}k''_l\Omega^2z)$ as done in Eq. (1.112), to obtain

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{A}_0 e^{-x\Omega^2} e^{iy\Omega^2} \quad (1.122)$$

where

$$x = \frac{\tau_{G0}^2}{4(1+a^2)} \quad (1.123)$$

and

$$y(z) = \frac{a\tau_{G0}^2}{4(1+a^2)} - \frac{k''_l z}{2}. \quad (1.124)$$

\tilde{A}_0 is a complex amplitude factor which we will not consider in what follows and τ_{G0} describes the pulse duration at the sample input. The time dependent electric field that we obtain by Fourier transforming Eq. (1.122) can be written as

$$\tilde{\mathcal{E}}(t, z) = \tilde{A}_1 \exp \left\{ - \left(1 + i \frac{y(z)}{x} \right) \left(\frac{t}{\sqrt{\frac{4}{x}[x^2 + y^2(z)]}} \right)^2 \right\}. \quad (1.125)$$

Obviously, this describes again a linearly chirped Gaussian pulse. For the ‘‘pulse duration’’ (note $\tau_p = \sqrt{2 \ln 2} \tau_G$) and phase at position z we find

$$\tau_G(z) = \sqrt{\frac{4}{x}[x^2 + y^2(z)]} \quad (1.126)$$