Analogy between pulse and beam propagation

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1 Time analogy of the paraxial approximation

Comparing the paraxial wave equation and the reduced wave equation describing pulse propagation through a GVD medium we notice an interesting correspondence. Both equations are of similar structure. In terms of the reduced wave equation the transverse space coordinates x, y seem to play the role of the time variable. This space-time analogy suggests the possibility of translating simply the effects related to dispersion into beam propagation properties. For instance, we may compare the temporal broadening of an unchirped pulse due to dispersion with the change of beam size due to diffraction. In this sense free-space propagation plays a similar role for the beam characteristics as a GVD medium does for the pulse envelope. The field spectrum at a distance z is:

$$\tilde{\mathscr{E}}(k_x, z) = \tilde{\mathscr{E}}(k_x, z=0)e^{i(k_x^2+k_y^2)z/(2k_\ell)}. \tag{1}$$

In time, the spectral envelope after propagation through a thickness z of a linear transparent material is given by:

$$\tilde{\mathscr{E}}(\Omega, z) = \tilde{\mathscr{E}}(\Omega, 0)e^{-\frac{i}{2}k_\ell''\Omega^2 z} \tag{2}$$

The exponential phase factor $(k_x^2 + k_y^2)z/(2k)$ which describes transverse beam diffraction in space, corresponds to the exponential phase factor $-k''\Omega^2 z/2$ which describes pulse dispersion in time. There is a correspondence $-k'' \to 1/k$. A difference that brings some complication is the dimensionality: 2 dimensions in space versus one dimension in time.

Since Eq. (1) corresponded to the paraxial approximation, the analogy can be carried over to successive subsets of that approximation. It will thus apply also to Gaussian optics, and the time equivalent of the Fraunhofer and geometric approximations.

2 Gaussian beams and Gaussian pulses

An important particular solution of the wave equation within the paraxial approximation is the Gaussian beam (see, e.g., [1]). In order to understand better the space-time analogy for Gaussian beams/pulses, the derivation is reproduced below.

2.1 Gaussian beams

We look for a solution to the wave equation:

$$\frac{\partial \tilde{\mathscr{E}}}{\partial z} + \frac{i}{2k}\left(\frac{\partial^2 \tilde{\mathscr{E}}}{\partial x^2} + \frac{\partial^2 \tilde{\mathscr{E}}}{\partial y^2}\right) = 0 \tag{3}$$

of the form:

$$\tilde{\mathscr{E}} = \mathscr{E}_0 e^{-i\left[P + \frac{k}{2q}(x^2+y^2)\right]} \tag{4}$$

Substituting (only the coefficients of the common exponentials are written below):

$$\frac{\partial \tilde{\mathscr{E}}}{\partial z} = -i\frac{dP}{dz} + i\frac{kr^2}{2q^2}\frac{dq}{dz}, \tag{5}$$

where we have written r^2 for $x^2 + y^2$.

$$\frac{\partial \tilde{\mathscr{E}}}{\partial x} = -i\frac{kx}{q}. \tag{6}$$

Taking the second derivative:

$$\frac{\partial^2 \tilde{\mathscr{E}}}{\partial x^2} + \frac{\partial^2 \tilde{\mathscr{E}}}{\partial y^2} = -2i\frac{k}{q} - \frac{k^2}{q^2}r^2. \tag{7}$$

The factor 2 in the first term of the right hand side is the main difference arise between the two dimensional space and the one dimensional time situations. Substituting in the wave Eq. (3), and equating the terms of same power in r:

$$i\frac{dP}{dz} = \frac{1}{q} \tag{8}$$

$$\frac{dq}{dz} = 1 \tag{9}$$

The last equation gives us:

$$q = q_0 + z \tag{10}$$

We separate $1/q$ in a real and imaginary part:

$$\frac{1}{q} = \frac{1}{R} - \frac{i}{\rho} \tag{11}$$

We choose the z axis such that at $z = 0$ $q_0 = i\rho_0$ is purely imaginary ($R = \infty$). Separating real and imaginary parts in Eq. (10):

$$R = R(z) = z + \rho_0^2/z \tag{12}$$

$$\rho = \rho_0(1 + z^2/\rho_0^2) \tag{13}$$

$$\tag{14}$$

Equation (8) can be integrated, leading to

$$P = -i\ln\left(\frac{q_0 + z}{q_0}\right)$$

and

$$e^{-iP} = \frac{q_0}{q_0 + z} = \frac{1}{1 - i\frac{z}{\rho_0}} = \frac{1}{\sqrt{1 + \frac{z^2}{\rho_0^2}}}e^{-\Phi(z)}, \tag{15}$$

where Φ is the Guoy phase shift,

$$\Theta = \Theta(z) = \arctan(z/\rho_0). \tag{16}$$

Up to Eq. (15), everything can be transposed to the time domain, with the parameters having the *same dimension*.

2.2 Gaussian Pulses

We look for a solution to the wave equation:

$$\frac{\partial \tilde{\mathscr{E}}}{\partial z} - \frac{ik''}{2}\left(\frac{\partial^2 \tilde{\mathscr{E}}}{\partial t^2}\right) = 0 \tag{17}$$

of the form:

$$\tilde{\mathscr{E}} = \mathscr{E}_0 e^{-i\left[Q + \frac{1}{2p}(t^2)\right]} \tag{18}$$

Substituting (only the coefficients of the common exponentials are written below):

$$\frac{\partial \tilde{\mathscr{E}}}{\partial z} = -i\frac{dQ}{dz} + i\frac{t^2}{2p^2}\frac{dp}{dz}, \tag{19}$$

where we have now t^2 instead of $x^2 + y^2$.

$$\frac{\partial \tilde{\mathscr{E}}}{\partial t} = -i\frac{t}{p}. \tag{20}$$

Taking the second derivative:

$$\frac{\partial^2 \tilde{\mathscr{E}}}{\partial t^2} = -\frac{i}{p} - \frac{t^2}{p^2} \tag{21}$$

The factor 2 in the first term of the right hand side is no longer there because of the difference in dimensionality. Substituting in the wave Eq. (17), and equating the terms of same power in t:

$$\frac{dQ}{dz} = i\frac{k''}{2p} \tag{22}$$

$$\frac{dp}{dz} = -k'' \tag{23}$$

$$\tag{24}$$

The last equation gives us:

$$p = p_0 + |k''|z \tag{25}$$

There is here a fundamental difference between space and time optics. In vacuum and in a linear dielectric, a beam can only expand by diffraction according to Eq. (10). In the time domain, dispersion can take a positive (normal dispersion) or negative value. However, no matter the sign of dispersion, p_0 (as well as q_0 in space) is chosen to start at a minimum positive value at $z = 0$. Therefore, the term $|k''|z$ has to be chosen positive. We separate $1/p$ in a real and imaginary part:

$$\frac{1}{p} = \frac{1}{R} - \frac{i}{\sigma} \tag{26}$$

We choose the z axis such that at $z = 0$ $p_0 = i\sigma_0$ is purely imaginary ($R = \infty$). Separating real and imaginary parts in Eq. (25):

$$R = |k''|z + \sigma_0^2/(|k''|z) \tag{27}$$

$$\sigma = \sigma_0[1 + (|k''|z)^2/\sigma_0^2] \tag{28}$$

$$\tag{29}$$

Equation (22) can be integrated, leading to

$$2Q = -i \ln\left(\frac{p_0 + |k''|z}{p_0}\right)$$

and

$$e^{-iQ} = \sqrt{\frac{p_0}{p_0 + |k''|z}} = \frac{1}{\sqrt{1 + i\frac{|k''|z}{\sigma_0}}} = \frac{1}{\left(1 + \frac{(|k''|z)^2}{\sigma_0^2}\right)^{1/4}} e^{-\Theta(z)/2} = \sqrt{\frac{\tau_{G0}}{\tau_G}} e^{-\Theta(z)/2}, \tag{30}$$

where Φ is the Guoy phase shift,

$$\Theta = \Theta(z) = \arctan(|k''|z\sigma_0). \tag{31}$$

Now we have to define R and σ in terms of the pulse duration and phase. If we choose:

$$\sigma = \frac{\tau_G^2}{2}, \tag{32}$$

the propagation Eq. (25) is satisfied. σ is therefore a positive, with a minimum positive value, hence the choice of sign in Eq. (25). For the focal distance associated with a phase modulation:

$$R = \frac{1}{\ddot{\varphi}}. \tag{33}$$

Replacing Eq. (32) and (33) in Eq. (18) we find indeed the correct temporal dependence of the Gaussian amplitude and phase.

2.3 "*p*" complex parameter, time correspondent of the spatial "q" parameter

A convenient quantity, labeled the q parameter, has been defined for Gaussian beams. It concatenates the information on the beam radius w and the radius of curvature R in a single complex quantity defined by:

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi w^2} \tag{34}$$

It has been observed that the modification of the q parameter by an optical element can be expressed in terms of the elements of the $ABCD$ matrix:

$$\frac{1}{q_2} = \frac{C + D/q_1}{A + B/q_1} \tag{35}$$

where q_1 and q_2 represent the value of the q parameter before and after the optical element, respectively. Equation (35) is often represented in the form:

$$\begin{pmatrix} q_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ 1 \end{pmatrix}. \tag{36}$$

For equivalence with Eq. (35), a re-normalization of the 'q' vector is needed after the matrix multiplication.

An example of application of ABCD matrices in space is the study of cavity stability. For a cavity characterized by an ABCD matrix, the evolution of the 'q' parameter over N round trips is given by:

$$\begin{pmatrix} q_N \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} q_{N-1} \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{N-1} \cdot \begin{pmatrix} q_1 \\ 1 \end{pmatrix} \tag{37}$$

It can be shown [1] that, for the beam to be trapped in the cavity, there is a stability condition: $-1 \leq \frac{1}{2}(A + D) \leq 1$.

The time equivalent of the q parameter:

$$\frac{1}{p} = \ddot{\varphi} - \frac{2i}{\tau_G^2}, \tag{38}$$

where $\ddot{\varphi} = \frac{\partial^2 \varphi}{\partial t^2}$ is the second derivative of the phase in the middle of the pulse, and τ_G [remembering that $\tau_p = \sqrt{2\ln 2}\tau_G$] is the Gaussian pulse width. The matrices for dispersion and time lensing as defined in Eq. (47) can be applied.

3 From spatial optics to temporal optics

3.1 ABCD matrix in space

An ABCD matrix [1] is a ray transfer matrix which describes the effect of an optical element on a laser beam. It can be used both in geometrical optics and for propagating Gaussian beams. The paraxial approximation is always required for ABCD matrix calculations. Tracing of a light path through an optical system can then be performed by multiplying an element matrix by a vector representing the light ray:

$$\begin{pmatrix} y_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix} \tag{39}$$

where y and α refer to transverse displacement and offset angle from an optical axis respectively. The subscripts '1' and '2' denote the coordinates before and after an optical element. For example, a thin lens with focal length f has the following ABCD matrix:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \tag{40}$$

and propagation through free space over a distance d is associated with the matrix:

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{41}$$

3.2 Transition from space to time

Geometric optics can be seen as an approximation of Gaussian propagation, where the propagation distance is much larger than the Rayleigh range. Therefore, the propagation in a linear medium is in a straight line making an angle α with the optics axis:

$$w = w_0\sqrt{1+\left(\frac{z}{\rho_0}\right)^2} \approx \frac{w_0}{\rho_0}z = \alpha z = \frac{2}{k_\ell w_0}z. \tag{42}$$

The same approximation can be made in the time domain to define the "optical inclination" α_t:

$$\tau = \tau_0\sqrt{1+\left(\frac{z}{L_d}\right)^2} \approx \frac{\tau_0}{L_d}z = \alpha_t z = \frac{2|k''|_\ell}{\tau_{G0}}z. \tag{43}$$

Note that in contrast to the space domain where α is dimensionless, α_t has the dimension of the inverse of a velocity. The generic operation of Eq. (39) has its time domain equivalent:

$$\begin{pmatrix} T_2 \\ \alpha_{t,2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ \alpha_{t,1} \end{pmatrix} \tag{44}$$

where T, the correspondent of the transverse displacement y, is a temporal position. In the time equivalent of the propagation matrix (41), the distance d is replaced by $|k''|_\ell d$, while the time equivalent of the lens matrix (40) has the element $-1/f$ replaced by an imposed chirp $\ddot{\Phi}$ which, for instance, in the case of Kerr modulation, is equal to:

$$\ddot{\Phi} = \frac{2\pi\ell_{Kerr}}{\lambda} n_2 \frac{I}{\tau_G^2}, \tag{45}$$

where ℓ_{Kerr} is the length of the nonlinear medium characterized by an intensity dependent index $n_2 I$. The matrix representation of the imaging problem of a standard imaging problem in space, is:

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix}, \tag{46}$$

while the time correspondent is:

$$\begin{pmatrix} T' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & |k''|_2 d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\ddot{\Phi} & 1 \end{pmatrix} \begin{pmatrix} 1 & |k''|_1 d_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T \\ \alpha \end{pmatrix} \tag{47}$$

The various level of approximation of space-time analogy are summarized in Table 1.

3.3 Beam widening by diffraction versus pulse broadening (in time) by dispersion

A quadratic phase factor in the propagation equation in a dispersive medium broadens an unchirped input pulse and leads to a (linear) frequency sweep across the pulse (chirp) while the pulse spectral width (and shape) remains unchanged. In an analogous manner we can interpret the beam profile widening due to diffraction. A "bandwidth-limited" Gaussian beam means a beam without phase variation across the beam, which requires a radius of curvature of the phase front $R = \infty$. Thus, a Gaussian beam is "bandwidth-limited" at its waist where it takes on its minimum possible size (at a given spatial frequency spectrum). Multiplication with a quadratic phase factor to describe the beam propagation leads to beam broadening and "chirp." The latter simply accounts for a finite phase front curvature. Roughly speaking, the spatial frequency components which are no longer needed to form the broadened beam profile are responsible for the beam divergence. Table 2 summarizes a comparison of the characteristics of Gaussian beam and pulse propagation.

Space		Time
$\left(\frac{\partial^2}{\partial x^2} - 2ik_\ell \frac{\partial}{\partial z}\right) \mathscr{E}(x,z) = 0$		$\left(\frac{\partial}{\partial z} + \frac{n}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{n}{c}\frac{\partial}{\partial t}\right) \mathscr{E}(\Omega,z) = 0$
$\left(\frac{\partial}{\partial z} - \frac{ik_x^2}{2k_\ell}\right) \tilde{u}(x,z) = 0$	Fourier transform	$\left(\frac{\partial}{\partial z} + ik_\ell(\Omega)\right) \mathscr{E}(\Omega,z) = 0$
$\mathscr{E}(k_x,z) = \mathscr{E}(k_x,0) \exp\left(\frac{i}{2k_\ell}k_x^2 z\right)$		$\mathscr{E}(\Omega,z) = \mathscr{E}(\Omega,0) \exp\left(-\frac{i\lvert k''\rvert_\ell}{2}\Omega^2 z\right)$
$\frac{k_x^2}{2k_\ell}$	$\Longleftrightarrow$	$-\frac{\lvert k''\rvert_\ell \Omega^2}{2}$
$\tilde{u}(x,z) \propto \int_{-\infty}^{\infty} \tilde{u}(x_0,0) \exp\left(i\frac{k_\ell x}{L}x_0\right) dx_0$	Fraunhofer approximation	$\tilde{\mathscr{E}}(t,z) \propto \int_{-\infty}^{\infty} \tilde{\mathscr{E}}(t_0,0) \exp\left(i\frac{t}{\lvert k''\rvert_\ell z}t_0\right) dt_0$
$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$	Displacement matrix	$\begin{pmatrix} 1 & \lvert k''\rvert_\ell\, d \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$	Lens matrix	$\begin{pmatrix} 1 & 0 \\ \ddot{\varphi} & 1 \end{pmatrix}$

Table 1: Space-time equivalence, starting from the Fourier transform of Maxwell's equation in space (left) and in time (right).

3.3.1 Application to a simple mode-locked cavity – stability criterium

To further clarify the time-space analogy, we consider a simple mode-locked laser cavity as sketched in Fig. 1. The time equivalent of the "flat mirror – curved mirror (radius R)" cavity of length L [Fig. 1 (a)], is one that starts from a bandwidth limited pulse at one end, propagates through the dispersion ($\lvert k''\rvert L$) of the cavity, goes through a Kerr self phase modulation ($\ddot{\Phi}$), then dispersion again to the starting point [Fig. 1 (b)]. The ABCD matrix for this cavity is, in space:

$$\begin{pmatrix} 1 - \frac{2}{R}L & 2L(1-\frac{2}{R}L) \\ -4\frac{L}{R} & 1-\frac{2}{R}L \end{pmatrix} \tag{48}$$

Gaussian pulse	**Gaussian beam**
bandwidth-limited pulse at $z=0$ (unchirped pulse)	beam waist at $z=0$ (plane phase fronts)
$\tilde{\mathscr{E}}_0(t) \propto e^{-(t/\tau_{G0})^2}$ $\tilde{\mathscr{E}}_0(\Omega) \propto e^{-(\tau_{G0}\Omega/2)^2}$	$\tilde{u}_0(x) \propto e^{-(x/w_0)^2}$ $\tilde{u}_0(k_x) \propto e^{-(k_x w_0/2)^2}$
Propagation through a medium of length L (dispersion)	Free space propagation over distance L (diffraction)
$\tilde{\mathscr{E}}(\Omega,L) \propto \exp\left[-\left(\frac{\tau_{G0}\Omega}{2}\right)^2 - i\frac{k_\ell'' L\Omega^2}{2}\right]$ $\tilde{\mathscr{E}}(t,L) \propto \exp\left[-(1+i\bar{a})\left(\frac{t}{\tau_G}\right)^2\right]$ $\propto \exp\left[i\frac{t^2}{2\tilde{p}(L)}\right]$ $\bar{a} = L/L_d$ $\tau_G(L) = \tau_{G0}\sqrt{1+\bar{a}^2}$	$\tilde{u}(k_x,L) \propto \exp\left[-\left(\frac{w_0 k_x}{2}\right)^2 + i\frac{L{k_x}^2}{2k_\ell}\right]$ $\tilde{u}(x,L) \propto \exp\left[-(1+i\bar{b})\left(\frac{x}{w}\right)^2\right]$ $\propto \exp\left[-ik_\ell\frac{x^2}{2\tilde{q}(L)}\right]$ $\bar{b} = L/\rho_0$ $w(L) = w_0\sqrt{1+\bar{b}^2}$
Chirp coefficient (slope)	Wavefront curvature
$\ddot{\varphi} = \frac{2\bar{a}}{1+\bar{a}^2}\,\frac{1}{\tau_{G0}^2}$	$\frac{1}{R} = \frac{\bar{b}}{1+\bar{b}^2}\,\frac{1}{\rho_0}$
Characteristic (dispersion) length	Characteristic (Rayleigh) length
$L_d = \frac{\tau_{G0}^2}{2k_\ell''}$	$\rho_0 = \frac{n\pi w_0^2}{\lambda_\ell} = \frac{k_\ell w_0^2}{2}$
Complex pulse parameter	Complex beam parameter
$\frac{1}{\tilde{p}(L)} = \ddot{\varphi}(L) + \frac{2i}{\tau_G^2(L)}$	$\frac{1}{\tilde{q}(L)} = \frac{1}{R(L)} + \frac{i\lambda_\ell}{\pi w^2(L)}$

Table 2: Comparison of dispersion and diffraction

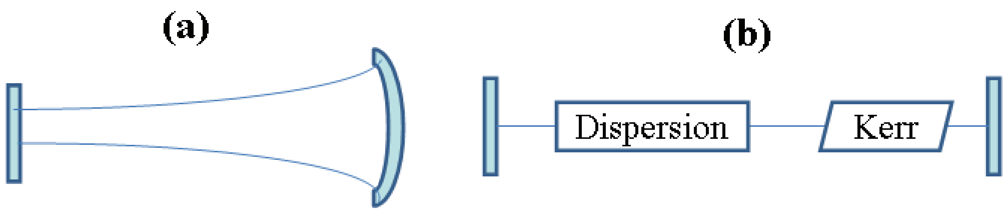

Figure 1: Example of s simple laser cavity in space (a) and in time (b). The round-trip matrix start at the flat mirror (before the dispersion in (b).

and in time:

$$\begin{pmatrix} 1+2|k''|\ell\ddot{\Phi} & 2|k''|\ell(1+|k''|\ell\ddot{\Phi}) \\ 2\ddot{\Phi} & 1+2|k''|\ell\ddot{\Phi} \end{pmatrix} \tag{49}$$

Stable operation of the laser requires that the q or p parameter is equal to itself after a round-trip [1] leading to the solution:

$$\frac{1}{p} = \frac{D-A}{2B} \mp \frac{i}{2B}\sqrt{4-(A+D)^2}. \tag{50}$$

For a real solution to exist, the stability criterium is:

$$(A+D)^2 < 4 \tag{51}$$

which for the space cavity implies $R \leq \infty$ and $R \geq L$. The latter limit gives the smallest beam waist at the flat mirror (concentric cavity). For the time cavity, the stability criterium is $-2 < \ddot{\Phi}|k''|L < 0$, which implies opposite sign for the phase modulation and dispersion. The minimum pulse in the time cavity is given by:

$$\tau_G^2 = 4B\sqrt{4-(A+D)^2} = 16|k''|L(1+\ddot{\Phi}|k''|L)\sqrt{2\ddot{\Phi}|k''|L-(\ddot{\Phi}|k''|L)^2}. \tag{52}$$

The shortest pulse duration is achieved for the “time concentric” cavity with $\ddot{\Phi}|k''|L = -2$.

References

[1] H. W. Kogelnik and T. Li. Laser beams and resonators. *Appl. Opt.*, 5:1550–1567, 1966.