Time-Space matrices for gaussian beams

1 Review of ABCD matrix in space

An ABCD matrix [1] is a ray transfer matrix which describes the effect of an optical element on a laser beam. It can be used both in geometrical optics and for propagating Gaussian beams. The paraxial approximation is always required for ABCD matrix calculations. Tracing of a light path through an optical system can then be performed by multiplying an element matrix by a vector representing the light ray:

\[
\begin{pmatrix}
y_2 \\
\alpha_2
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
y_1 \\
\alpha_1
\end{pmatrix}
\] (1)

where \(y\) and \(\alpha\) refer to transverse displacement and offset angle from an optical axis respectively. The subscripts ‘1’ and ‘2’ denote the coordinates before and after an optical element. For example, a thin lens with focal length \(f\) has the following ABCD matrix:

\[
\begin{pmatrix}
1 & 0 \\
-1/f & 1
\end{pmatrix},
\] (2)

and propagation through free space over a distance \(d\) is associated with the matrix:

\[
\begin{pmatrix}
1 & d \\
0 & 1
\end{pmatrix}
\] (3)

Transition from space to time Geometric optics can be seen as an approximation of Gaussian propagation, where the propagation distance is much larger than the Rayleigh range. Therefore, the propagation in a linear medium is in a straight line making an angle \(\alpha\) with the optics axis:

\[
w = w_0 \sqrt{1 + \left(\frac{z}{\rho_0}\right)^2} \approx \frac{w_0}{\rho_0} z = \alpha z = \frac{2}{k^0 L} z.
\] (4)

The same approximation can be made in the time domain to define the “optical inclination” \(\alpha_t\):

\[
\tau = \tau_0 \sqrt{1 + \left(\frac{z}{L_d}\right)^2} \approx \frac{\tau_0}{L_d} z = \alpha_t z = \frac{2k^0}{\tau G_0} z.
\] (5)

Note that in contrast to the space domain where \(\alpha\) is dimensionless, \(\alpha_t\) has the dimension of the inverse of a velocity. The generic operation of Eq. (1) has its time domain equivalent:

\[
\begin{pmatrix}
T_2 \\
\alpha_{t,2}
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
T_1 \\
\alpha_{t,1}
\end{pmatrix}
\] (6)

where \(T\), the correspondent of the transverse displacement \(v\), is a temporal position. In the time equivalent of the propagation matrix (3), the distance \(d\) is replaced by \(k^0_d\), while the time equivalent of the lens matrix (2) has the element \(-1/f\) replaced by an imposed chirp \(\Phi\) which, for instance, in the case of Kerr modulation, is equal to:

\[
\Phi = \frac{2\pi k_{Kerr}}{\lambda n_2} \frac{L}{\tau^2},
\] (7)
where $\ell_{Kerr}$ is the length of the nonlinear medium characterized by an intensity dependent index $n_2 I$. The matrix representation of the imaging problem of a standard imaging problem in space, is:

$$
\begin{pmatrix}
\gamma' \\
\alpha'
\end{pmatrix} =
\begin{pmatrix}
1 & d_2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix}
\begin{pmatrix}
1 & d_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\gamma \\
\alpha
\end{pmatrix},
$$

(8)

while the time correspondent is:

$$
\begin{pmatrix}
T' \\
\alpha'
\end{pmatrix} =
\begin{pmatrix}
1 & k'' d_2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi & 1
\end{pmatrix}
\begin{pmatrix}
1 & k' d_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix} T \\
\alpha
\end{pmatrix}
$$

(9)

The various level of approximation of space-time analogy are summarized in Table 1.

<table>
<thead>
<tr>
<th>Space</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{\partial^2}{\partial x^2} - 2ik_\ell \frac{\partial}{\partial z} \right) \mathcal{E}(x,z) = 0 )</td>
<td>Fourier transform ( \left( \frac{\partial}{\partial z} + \frac{n}{c} \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial z} - \frac{n}{c} \frac{\partial}{\partial t} \right) \mathcal{E}(\Omega,z) = 0 )</td>
</tr>
<tr>
<td>( \left( \frac{\partial}{\partial z} - \frac{ik_\ell^2}{2k_\ell} \right) \tilde{u}(x,z) = 0 )</td>
<td>( \left( \frac{\partial}{\partial z} + ik_\ell(\Omega) \right) \mathcal{E}(\Omega,z) = 0 )</td>
</tr>
<tr>
<td>( \mathcal{E}(k_x,z) = \mathcal{E}(k_x,0) \exp\left( \frac{i k_\ell^2}{2k_\ell} z \right) )</td>
<td>( \mathcal{E}(\Omega,z) = \mathcal{E}(\Omega,0) \exp\left( -\frac{ik_\ell''}{2} \Omega^2 z \right) )</td>
</tr>
<tr>
<td>( \frac{k_x^2}{2k_\ell} )</td>
<td>( -\frac{k_\ell'' \Omega^2}{2} )</td>
</tr>
<tr>
<td>( \tilde{u}(x,z) \propto \int_{-\infty}^{\infty} \tilde{u}(x_0,0) \exp\left( i \frac{k_\ell x}{L} x_0 \right) dx_0 )</td>
<td>Fraunhofer approximation ( \tilde{E}(t,z) \propto \int_{-\infty}^{\infty} \tilde{E}(t_0,0) \exp\left( i \frac{t}{k_\ell'' \Omega^2} t_0 \right) dt_0 )</td>
</tr>
<tr>
<td>( \begin{pmatrix} 1 &amp; d \ 0 &amp; 1 \end{pmatrix} )</td>
<td>Displacement matrix ( \begin{pmatrix} 1 &amp; k'' d \ 0 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>( \begin{pmatrix} 1 &amp; 0 \ -\frac{1}{f} &amp; 1 \end{pmatrix} )</td>
<td>Lens matrix ( \begin{pmatrix} 1 &amp; 0 \ \Phi &amp; 1 \end{pmatrix} )</td>
</tr>
</tbody>
</table>

Table 1: Space-time equivalence, starting from the Fourier transform of Maxwell’s equation in space (left) and in time (right).
Table 2: Comparison of dispersion and diffraction

<table>
<thead>
<tr>
<th>Gaussian pulse</th>
<th>Gaussian beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>bandwidth-limited pulse at ( z = 0 ) (unchirped pulse)</td>
<td>beam waist at ( z = 0 ) (plane phase fronts)</td>
</tr>
<tr>
<td>( \tilde{E}_0(t) \propto e^{-(t/\tau_G)^2} )</td>
<td>( \tilde{u}_0(x) \propto e^{-(x/w_0)^2} )</td>
</tr>
<tr>
<td>( \tilde{E}_0(\Omega) \propto e^{-(\Omega \tau_G/2)^2} )</td>
<td>( \tilde{u}_0(k_x) \propto e^{-(k_x w_0/2)^2} )</td>
</tr>
<tr>
<td>Propagation through a medium of length ( L ) (dispersion)</td>
<td>Free space propagation over distance ( L ) (diffraction)</td>
</tr>
<tr>
<td>( \tilde{E}(\Omega, L) \propto \exp \left[ -\left( \frac{\Omega \tau_G}{2} \right)^2 - i \frac{k_y^2 \Omega^2 L}{2} \right] )</td>
<td>( \tilde{u}(k_x, L) \propto \exp \left[ -\left( \frac{w_0 k_x}{2} \right)^2 + i \frac{L k_x^2}{k}\right] )</td>
</tr>
<tr>
<td>( \tilde{E}(t, L) \propto \exp \left[ -\left( 1 + i \tilde{a} \right) \left( \frac{t}{\tau_G} \right)^2 \right] \propto \exp \left[ i \tilde{a} \frac{t^2}{2 \tilde{p}(L)} \right] )</td>
<td>( \tilde{u}(x, L) \propto \exp \left[ -\left( 1 + i \tilde{b} \right) \left( \frac{x}{w} \right)^2 \right] \propto \exp \left[ -i k \frac{x^2}{2 \tilde{q}(L)} \right] )</td>
</tr>
<tr>
<td>( \tilde{a} = \frac{L}{L_d} )</td>
<td>( \tilde{b} = \frac{L}{\rho_0} )</td>
</tr>
<tr>
<td>( \tau_G(L) = \tau_G \sqrt{1 + \tilde{a}^2} )</td>
<td>( w(L) = w_0 \sqrt{1 + \tilde{b}^2} )</td>
</tr>
<tr>
<td>Chirp coefficient (slope)</td>
<td>Wavefront curvature</td>
</tr>
<tr>
<td>( \phi = \frac{2 \tilde{a}}{1 + \tilde{a}^2} \frac{1}{\tau_G} )</td>
<td>( \frac{1}{R} = \frac{\tilde{b}}{1 + \tilde{b}^2} \frac{1}{\rho_0} )</td>
</tr>
<tr>
<td>Characteristic (dispersion) length</td>
<td>Characteristic (Rayleigh) length</td>
</tr>
<tr>
<td>( L_d = \frac{\tau_G^2}{2 k} )</td>
<td>( \rho_0 = \frac{n \pi w_0^2}{\lambda} = \frac{k \ell w_0^2}{2} )</td>
</tr>
<tr>
<td>Complex pulse parameter</td>
<td>Complex beam parameter</td>
</tr>
<tr>
<td>( \frac{1}{\tilde{p}(L)} = \phi(L) + \frac{2i}{\tau_G^2(L)} )</td>
<td>( \frac{1}{\tilde{q}(L)} = \frac{1}{R(L)} + \frac{i \lambda}{\pi w^2(L)} )</td>
</tr>
</tbody>
</table>

2 Gaussian pulses as analogue of Gaussian beams

A quadratic phase factor in the propagation equation in a dispersive medium broadens an unchirped input pulse and leads to a (linear) frequency sweep across the pulse (chirp) while the pulse spectral width (and shape) remains unchanged. In an analogous manner we can interpret the Fresnel integral for the beam profile. A “bandwidth-limited” Gaussian beam means a beam without phase variation across the beam, which requires a radius of curvature of the phase front \( R = \infty \). Thus, a Gaussian beam is “bandwidth-limited” at its waist where it takes on its minimum possible size (at a given spatial frequency spectrum). Multiplication with a quadratic phase factor to describe the beam propagation leads to beam broadening and “chirp.” The latter simply accounts for a finite phase front curvature. Roughly speaking, the spatial frequency components which are no longer needed to form the broadened beam profile are responsible for the beam divergence. Table 2 summarizes our discussion comparing the characteristics of Gaussian beam and pulse propagation.
3 Time-space analogy applied to cavity calculations

3.1 “p” complex parameter, time correspondent of the spatial “q” parameter

A convenient quantity, labeled the q parameter, has been defined for Gaussian beams. It concatenates the information on the beam radius \( w \) and the radius of curvature \( R \) in a single complex quantity defined by:

\[
\frac{1}{q} = \frac{1}{R} - i \frac{\lambda}{\pi w^2} \tag{10}
\]

It has been observed that the modification of the q parameter by an optical element can be expressed in terms of the elements of the \( ABCD \) matrix:

\[
\frac{1}{q_2} = \frac{C + D}{A + B} \frac{q_1}{q_1} \tag{11}
\]

where \( q_1 \) and \( q_2 \) represent the value of the q parameter before and after the optical element, respectively. Equation (11) is often represented in the form:

\[
\begin{pmatrix} q_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ 1 \end{pmatrix} \tag{12}
\]

For equivalence with Eq. (11), a re-normalization of the ‘q’ vector is needed after the matrix multiplication.

An example of application of \( ABCD \) matrices in space is the study of cavity stability. For a cavity characterized by an \( ABCD \) matrix, the evolution of the ‘q’ parameter over N round trips is given by:

\[
\begin{pmatrix} q_N \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_{N-1} \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{N-1} \begin{pmatrix} q_1 \\ 1 \end{pmatrix} \tag{13}
\]

It can be shown [1] that, for the beam to be trapped in the cavity, there is a stability condition: \(-1 \leq \frac{1}{2}(A + D) \leq 1\).

The time equivalent of the q parameter:

\[
\frac{1}{p} = \dddot{\phi} - \frac{2i}{\tau_G^2}, \tag{14}
\]

where \( \dddot{\phi} = \frac{\phi'}{\tau_p^2} \) is the second derivative of the phase in the middle of the pulse, and \( \tau_G \) [remembering that \( \tau_p = \sqrt{2\ln 2\tau_G} \)] is the Gaussian pulse width. The matrices for dispersion and time lensing as defined in Eq. (9) can be applied.
References