Complex electric field:

$$\tilde{E}^{+}(t) = \frac{1}{2\pi}\int_{0}^{\infty} \tilde{E}(\Omega) e^{i\Omega t} d\Omega \tag{1}$$

Spectral field strength with only positive frequencies:

$$\tilde{E}^{+}(\Omega) = |\tilde{E}(\Omega)| e^{i\Phi(\Omega)} = \begin{cases} \tilde{E}(\Omega) & \text{for} \quad \Omega \geq 0 \\ 0 & \text{for} \quad \Omega < 0 \end{cases} \tag{2}$$

$\tilde{E}^{+}(t)$ and $\tilde{E}^{+}(\Omega)$ are related to each other through:

$$\tilde{E}^{+}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}^{+}(\Omega) e^{i\Omega t} d\Omega \tag{3}$$

and

$$\tilde{E}^{+}(\Omega) = \int_{-\infty}^{\infty} \tilde{E}^{+}(t) e^{-i\Omega t} dt. \tag{4}$$

These quantities relate to the real electric field:

$$E(t) = \tilde{E}^{+}(t) + \tilde{E}^{-}(t) \tag{5}$$

and its complex Fourier transform:

$$\tilde{E}(\Omega) = \tilde{E}^{+}(\Omega) + \tilde{E}^{-}(\Omega) \tag{6}$$

The complicated but "sexy" approach

The Hilbert Transform

<u>**Reference**</u> (Textbook): S. Haykin, Communication System, 4th ed. (Wiley, New York, 2001).

The Hilbert transform correspond to applying a phase shift of $\pi/2$ to the components of a signal. The Hilbert transform of $g(t)$ is:

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{(t-\tau)} d\tau. \tag{7}$$

The inverse Hilbert transform:

$$g(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(\tau)}{(t-\tau)} d\tau. \tag{8}$$

These are convolutions of $g(\tau)$ with $1/(\pi\tau)$. The Fourier transform of $1/(\pi\tau)$ is the "signum function" (Fig. 1) corresponding indeed to a phase shift of $\pi/2$ between positive and negative frequencies. $\hat{g}(t)$ is also defined as the analytical continuation of the function $g(t)$. Given a real function $g(t)$, one defines the complex "analytic signal" $g_+(t)$ of $g(t)$ by:

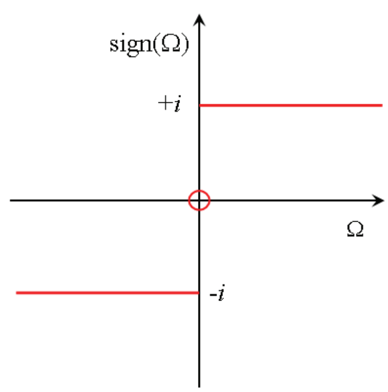

Figure 1: signum function.

$$g_+(t) = g(t) + i\hat{g}(t), \tag{9}$$

of which the Fourier transform is:

$$G_+(\Omega) = G(\Omega) + \mathrm{sign}(\Omega)\tilde{E}(\Omega). \tag{10}$$

The transformation from $G(\Omega)$ to the complex function $G_+(\Omega)$ corresponds to eliminating the negative part of the Fourier transform by adding its opposite. This is exactly how $\tilde{E}(t)$ was defined in the previous slide.

$G_+(\Omega) = 2\tilde{E}^+(\Omega)$ as defined in Eq. (6)
$g_+(t) = 2\tilde{E}^+(t)$ as defined in Eq. (5)
$g(t) = E(t)$

This approach was re-introduced "as something new" in nonlinear optics by Conforti *et al.* [1] and cited by many [2, 3]. The context is that of defining a nonlinear polarization:

$$P(t) = \chi^{(2)} E_s(t) E_i(t). \tag{11}$$

In complex notations, one often writes:

$$\tilde{P}(t) = \chi^{(2)} \tilde{E}_s(t) \tilde{E}_i(t). \tag{12}$$

Instead, Conforti *et al.* [1] define the complex $\tilde{P}(t)$ by taking the Fourier transform of $P(t)$ defined by Eq (11), eliminating the negative part, and taking the inverse Fourier transform. No need to evoke the Hilbert transform to perform that operation. The two approaches are equivalent when the spectra of the fields E_s and E_i do not overlap.

[1] M. Conforti, F. Baronio, and C. De Angelis. Nonlinear envelope equation for broadband optical pulses in quadratic media. *Physical Review A*, 81:053841, 2010.

[2] D. T. Reid. Ultra-broadband pulse evoluton in optical parametric oscillators. *Optics Express*, 19:17979–17984, 2011.

[3] A. S. Kowligy, A. Lind, D. Hickstein, D. R. Carlson, H. Timmers, N. Nader, F. C. Cruz, G.Ycas, S. B. Papp, and Scott A. Diddams. Mid-infrared frequency comb generation via cascaded quadratic nonlinearities in quasi-phase-matched waveguides. *Optics Letters*, 43:1678–1680, 2018.