What is the Confocal Parameter?

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Abstract—A novel route to Gaussian beams is presented. We start from spherical solutions to the three-dimensional Helmholtz equation. In this case, the usual substitution $z \to z + jb$ elegantly yields a new wave with ellipsoidal phase fronts. In the paraxial limit, this wave asymptotically approaches a Gaussian. Using this formulation, we recast resonator stability theory in a particularly straightforward way.

I. Introduction

The Gaussian beam is a useful representation of a focused wave of finite transverse dimensions. Although it has been extensively studied, many questions about the Gaussian remain active areas of research. A recent publication [1] illustrates one such open question: what is the relation of the Gaussian beam to the solutions of the three-dimensional wave equation? Discussed in this letter is a novel and enlightening method to derive the Gaussian as a limit of the solution to the full wave equation. As shall be seen, the confocal parameter takes on particular geometrical significance in this formulation.

Usually, the functional form of the Gaussian is found by way of a two-step process. First, one identifies the Green's function of the paraxial wave equation

$$h(x, y, z) = \frac{j}{\lambda z} e^{-jkz} \exp\left(\frac{-jk}{2z}(x^2 + y^2)\right). \quad (1)$$

Then, since the paraxial wave equation is invariant under translation $z \to z + z_o$, replacing z with $z + jb$ in (1) must also yield a legitimate solution [2]. The resulting solution is the familiar Gaussian beam

$$u_{oo} \propto \exp\left(-\frac{x^2 + y^2}{w^2}\right) \exp\left(-\frac{jk}{2R}(x^2 + y^2) - jkz\right) \quad (2)$$

where $w^2(z) = (2b/k)(1 + z^2/b^2)$ is the beamwidth, and $1/R(z) = z/(z^2 + b^2)$ is the radius of curvature of the phase fronts [2]. An alternate route to (2) is the introduction of a complex q-parameter [3], or "complex radius of curvature" [4] into (1). All of these approaches are equivalent.

Unfortunately, the physical meaning of translating the z axis by the imaginary distance jb is not at all clear (at least to this author). In fact, most authors regard this procedure as a purely formal mathematical exercise which (fortuitously) yields the correct answer. Here, it is shown for the first time that a deeper interpretation exists. We arrive at Gaussian beams by starting with a solution of the full three-dimensional Helmholtz equation in spherical coordinates. Then, performing the transformation $z \to z + jb$ corresponds physically to causing the phase fronts of the solution to become ellipsoids. (A similar geometry was used by Marcatili and Someda [1] to describe a focusing mode.) The separation of the foci of the ellipsoids is $2b$, where b is the confocal parameter of the beam. In the paraxial limit the new ellipsoidal solution becomes a Gaussian beam. Finally, we show that adopting this approach to Gaussian beams allows a simple, geometrical interpretation of the optical resonator stability criterion.

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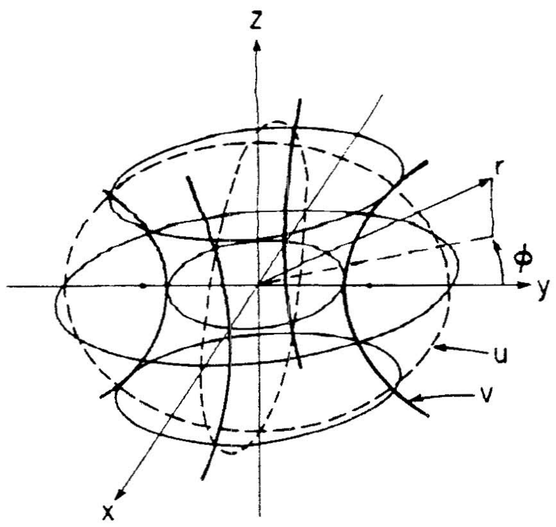
II. Ellipsoidal Phase Fronts

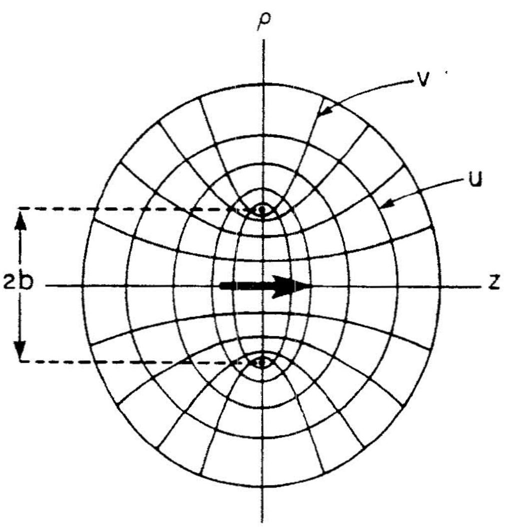
We begin by investigating the free-space solutions to the scalar Helmholtz equation, $\nabla^2\psi + k^2\psi = 0$, in spherical coordinates. In analogy to the paraxial wave equation, we desire the solution which corresponds to a source at the origin—that is, the Green's function. The Helmholtz equation is separable in these coordinates, and we have $\psi_n(r, \theta, \phi) = h_n^{(1)}(kr)\, Y_{l,m}(\theta, \phi)$, where $h_n^{(1)}(kr)$ is the spherical Hankel function of the first kind. Since $r = \sqrt{x^2 + y^2 + z^2}$, $h_n^{(1)}(kr)$ possesses spherical symmetry. Replacing $z \to z + jb$ breaks this symmetry, rendering spherical coordinates inappropriate for describing $h_n^{(1)}(kr)$. Now choose a new coordinate system defined by

$$x = b \cosh u \cos v \sin \phi$$
$$y = b \cosh u \cos v \cos \phi$$
$$z = b \sinh u \sin v. \quad (3)$$

This is the definition of the oblate spherical coordinate system [5]. [See Fig. 1(a).] This coordinate system is symmetric with respect to rotations around the z axis. Since we are interested in describing axially symmetric Gaussian beams, we may ignore the angular coordinate ϕ by only considering ψ on a plane containing the z axis. We take z as the axial coordinate, and $\rho = \sqrt{x^2 + y^2}$ as the radial coordinate, as depicted in Fig. 1(b). On this plane, transformation (3) describes a coordinate system consisting of a set of confocal ellipsoids parameterized by u (where $u \in \{0, \infty\}$), orthogonal at every point to a set of confocal parabaloids parameterized by v, (where $v \in \{0, \pi\}$) [5]. The foci of the ellipsoids lie at $\rho = \pm b$. [Fig. 1(b).]



(a)



(b)

Fig. 1. Coordinate systems used in describing ellipsoidal phase fronts. (a) Oblate spherical coordinates, parameterized by u (confocal ellipses), v (confocal hyperbolas), and ϕ (angular coordinate). The two dots on the y axis are the focal points of the ellipses and hyperbolas on the yz plane. The beam propagates in the $+z$ direction. (b) Elliptic coordinate system obtained by making a cross-sectional cut through (a) including the z axis. In this case, $\rho = \sqrt{x^2 + y^2}$. The foci lie at $\rho = \pm b$. The arrow indicates the direction of the beam propagation.

With this transformation, we may write

$$\tilde{r} \equiv \sqrt{\rho^2 + (z + jb)^2}$$
$$= b\sqrt{\cosh^2 u \cos^2 v + \sinh^2 u \sin^2 v - 1 + 2j \sinh u \sin v}.$$

Recalling $\cosh^2 u - \sinh^2 u = 1$, and $\sin^2 v + \cos^2 v = 1$, we may rewrite this expression as

$$\tilde{r} = b(\sinh u + j \sin v). \tag{4}$$

Thus we find that u and v correspond to the real and imaginary components of the argument of $h_n^{(1)}(k\tilde{r})$. Whereas previously $\tilde{r}$ was a real number, now points given by $v \neq 0$ or π give complex values of $\tilde{r}$. In this sense, fixing a value of u and varying v away from $v = 0$ corresponds to analytic continuation of $h_n^{(1)}(k\tilde{r})$.

In two dimensions, the lines of constant phase of the wave must be found by studying the behavior of $h_n^{(1)}(k\tilde{r})$ in the complex $\tilde{r}$ plane. For simplicity, we investigate $h_o^{(1)}(k\tilde{r})$, although the results obtained in this particular case carry through for all n. Recall the expression for $h_o^{(1)}(k\tilde{r})$ [5]

$$h_o^{(1)}(k\tilde{r}) = \frac{\exp(-jk\tilde{r})}{k\tilde{r}}. \tag{5}$$

We may ignore the small phase factor introduced by the prefactor $1/k\tilde{r}$. Then, the lines of constant phase are those for which

$$\mathrm{Re}\left\{k\tilde{r}\right\} = \text{constant}.$$

Clearly, since k is real, and the real part of $\tilde{r}$ is parameterized by u, we immediately have that varying v—the imaginary part of $\tilde{r}$—does not change the phase of the wave. Since fixing u and varying v corresponds to walking along an ellipse in the (u, v) plane, we see that in three dimensions the surfaces of constant phase are ellipsoids parameterized by u. Similar logic is true for all n.

III. Wave Equation Solutions

Because the solution (5) has been obtained from a simple displacement of the origin, $z \rightarrow z + jb$, $\psi(k\tilde{r})$ must still obey the Helmholtz equation. However, since the phase fronts of the wave are parameterized by the set of confocal ellipsoids, we see that $\psi(k\tilde{r})$ is no longer a solution in spherical coordinates, but must instead be a solution in oblate spheroidal coordinates. The nature of this solution is quite interesting.

We rewrite the transformation (3) using new variables

$$x = b\xi\eta \sin \phi$$
$$y = b\xi\eta \cos \phi$$
$$z = b\sqrt{(\xi^2 - 1)(1 - \eta^2)} \tag{6}$$

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where $\xi \in \{1, \infty\}$ parameterizes the ellipses, and $\eta \in \{-1, 1\}$ parameterizes the hyperbolas [5]. In these co-oblate spheroidal symmetry as long as ψ satisfies (10), i.e., ψ is a spherical Bessel (or Hankel) function.

IV. Relationship to Gaussian Beams

It is important to know the relationship of our solution (5) to the usual Gaussian beam. Not surprisingly, we find that under the paraxial approximation, (5) approaches a Gaussian beam. The derivation proceeds as follows. Using the coordinate system defined by (6), we can express (4) as

$$\tilde{r} = b(\sqrt{\xi^2 - 1} + j\sqrt{1 - \eta^2}). \tag{11}$$

If we assume paraxial rays, then $\xi >> 1$ and $|\eta| << 1$. In this situation, (6) becomes

$$\rho = b\xi\eta$$
$$z = b\xi(1 - \tfrac{1}{2}\eta^2) \tag{12}$$

where $\rho = \sqrt{x^2 + y^2}$. Similarly, we get for (11)

$$\tilde{r} = b(\xi + j(1 - \tfrac{1}{2}\eta^2)). \tag{13}$$

Now solving (12) approximately for $b\xi$ and $(1 - \frac{1}{2}\eta^2)$, we get

$$b\xi \approx z + \frac{1}{2}\frac{\rho^2}{z}$$
$$1 - \frac{1}{2}\eta^2 \approx 1 - \frac{1}{2}\frac{\rho^2}{z^2}.$$

Inserting these into (13) gives $\tilde{r}$, from which we obtain ordinates, the Helmholtz equation may be written [5]

$$\frac{\sqrt{\xi^2 - 1}}{\xi}\frac{\partial}{\partial\xi}\left[\xi\sqrt{\xi^2 - 1}\frac{\partial\psi}{\partial\xi}\right] + \frac{\sqrt{1 - \eta^2}}{\eta}\frac{\partial}{\partial\eta}\left[\eta\sqrt{1 - \eta^2}\frac{\partial\psi}{\partial\eta}\right] + \left(\frac{\xi^2 - \eta^2}{\xi^2\eta^2}\right)\frac{\partial^2\psi}{\partial\phi^2} + b^2k^2(\xi^2 - \eta^2)\psi = 0. \tag{7}$$

Utilizing the fact that ψ can be written $\psi(bk\sqrt{\xi^2 - 1} + jbk\sqrt{1 - \eta^2})$ allows us to write (7) as

$$b^2k^2(\xi^2 - \eta^2)\psi'' + 2bk(\sqrt{\xi^2 - 1} - j\sqrt{1 - \eta^2})\psi' + k^2b^2(\xi^2 - \eta^2)\psi = 0. \tag{8}$$

If we define

$$t = bk\sqrt{\xi^2 - 1} + jbk\sqrt{1 - \eta^2}$$

we can express (8) as

$$t^*t\psi''(t) + 2t^*\psi'(t) + t^*t\psi(t) = 0. \tag{9}$$

Dividing out by t^*t gives the equation for the zeroth-order spherical Bessel (or Hankel) function [5]:

$$\psi''(t) + \frac{2}{t}\psi'(t) + \psi(t) = 0. \tag{10}$$

Thus, we have shown that the replacement $z \to z + jb$ yields a solution for the Helmholtz equation possessing the new wave

$$\psi_o(\rho, z) \propto \exp\left(-\frac{bk}{2}\frac{\rho^2}{z^2}\right)\exp\left(-jkz - \frac{jk}{2z}\rho^2\right). \tag{14}$$

As can be seen, (14) is identical to (2) in the limit $z >> b$. Thus, we have shown that the Green's function $h_o^{(1)}(k\tilde{r})$ asymptotes to the Gaussian beam in the paraxial limit.

A great advantage of starting with the full Helmholtz equation to obtain Gaussian beams is that one may avoid using functions possessing singularities for finite z like (1). This is not possible when starting from the paraxial wave equation. Specifically, the solution $\psi_o = j_o(k\tilde{r})$ also reduces to (14) in the paraxial limit, but provides a superior description of a standing wave in a resonator since $j_o(k\tilde{r})$ is analytic for all finite $\tilde{r}$.

V. Resonator Stability

Typically, the stability of an optical resonator with curved mirrors is determined by fitting a Gaussian beam into the resonator, and matching the radius of curvature of the beam with the mirror radius of curvature [2]. If such a beam exists, and its confocal parameter is real, then the resonator is stable. Otherwise, the resonator is unstable. Using this approach, expressing b in terms of the resonator parameters R_1 and R_2 (mirror radii of curvature), and d (mirror separation) is a nonenlightening exercise in algebra. Happily, recognizing that the phase fronts of the beam are ellipsoidal allows a very simple, geometrical interpretation of the resonator stability criterion, achieving a considerable simplification in its derivation.

Since the beam phase fronts are ellipsoidal, the problem of building a stable resonator is equivalent to that of finding an elliptical coordinate system on which the mirrors have the same radii of curvature as the ellipses along the z axis, and are separated by the distance d. (See Fig. 2.) With these conditions, the only free parameters are the placement of the origin of the coordinate system, and the focal separation $2b$. The positions of the mirrors are expressible as

$$z_1 = b\sinh u_1$$
$$z_2 = b\sinh u_2. \tag{15}$$

while the radii of curvature of the mirrors along the z axis are

$$R_1 = b\frac{\cosh^2 u_1}{\sinh u_1}$$
$$R_2 = b\frac{\cosh^2 u_2}{\sinh u_2}. \tag{16}$$

Our task is to solve (15) and (16) for b subject to the constraint

$$d = z_1 + z_2. \tag{17}$$

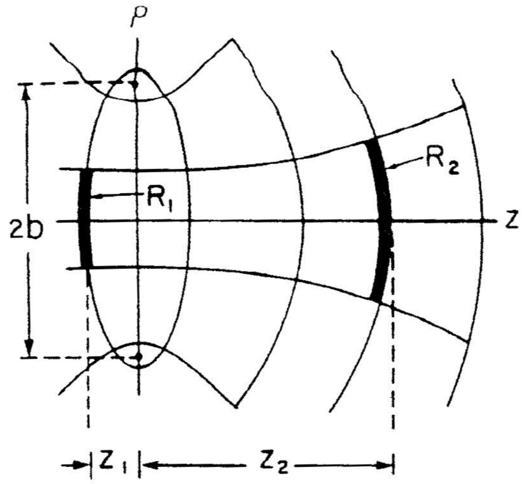

Fig. 2. Fitting a coordinate system to the resonator mirrors (R_1 and R_2). The two free parameters are the placement of the origin (fixing z_1 and z_2), and the focal separation $2b$. The constraints are the mirror curvatures R_1 and R_2 must match the curvature of the ellipses along the z axis, and the mirror separation $d = z_1 + z_2$ is fixed.

Solving for sinh u_1 in (16), we obtain

$$\sinh u_1 = \left(\frac{R_1}{2b}\right) \pm \sqrt{\frac{R_1^2}{4b^2} - 1}$$

and similiarly for sinh u_2. Imposing constraint (17) leads directly to the expression

$$d = \frac{R_1}{2} + \frac{R_2}{2} \pm \sqrt{\frac{R_1^2}{4} - b^2} \pm \sqrt{\frac{R_2^2}{4} - b^2}. \quad (18)$$

This must be satisfied by real b in order for the resonator to be stable. This expression is precisely the usual resonator stability criterion, and may be found in any undergraduate laser test [2], [4]. Remarkably, we have derived (18) without the usual algebraic clutter, in a manner that leaves the geometrical nature of resonator stability clearly revealed.

VI. Summary

To recapitulate, we have demonstrated that the transformation $z \rightarrow z + jb$ in a spherically symmetric solution of Helmholtz's equation breaks the spherical symmetry, yielding a solution possessing oblate spheroidal symmetry. The phase fronts of the resulting wave are ellipsoids. The new wave is a solution of Helmholtz's equation in oblate spherical coordinates, although the functional form of the solution is that of a wave in spherical coordinates. In the limit $z >> b$, the zeroth order solution $h_o^{(1)}(k\tilde{r})$ approaches a Gaussian beam. Finally, using this new coordinate system we can derive the usual resonator stability condition in a particularly elegant way. It is our hope that these results lay bare the essentially geometrical nature of the transformation $z \rightarrow z + jb$. To conclude, we may answer the question: what is the confocal parameter? It is simply one half the distance between focal points in the elliptic coordinate system describing the phase fronts of the Gaussian beam!

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