1

a.1 $E(t) = E_0 \,\mathrm{sech}(t/\tau)\, e^{+i\omega_0 t} = \dfrac{2E_0\, e^{+i\omega_0 t}}{e^{t/\tau}+e^{-t/\tau}}$

$\{\mathcal{F}\}$ $E(\omega) = \int_{-\infty}^{\infty} E(t)\, e^{-i\omega t}\, dt = 2E_0 \int_{-\infty}^{\infty} \dfrac{e^{-i(\omega-\omega_0)t}}{e^{t/\tau}+e^{-t/\tau}}\, dt$

Let $t = \mathrm{Re}[z]$

$$E(\omega) = 2E_0 \oint_c f(z)\, dz, \text{ where } f(z) = \frac{e^{-i(\omega-\omega_0)z}}{e^{z/\tau}+e^{-z/\tau}} \equiv \frac{q(z)}{g(z)}$$

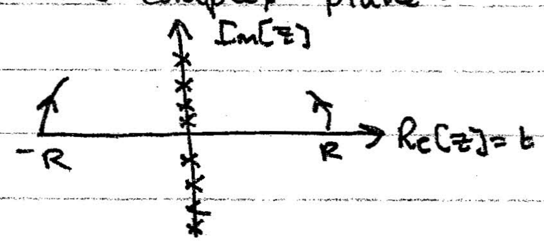
poles exist when $g(z) = e^{z/\tau}+e^{-z/\tau} = 0 \Rightarrow e^{2z/\tau}+1 = 0$

$$e^{2z_{0n}/\tau} = -1 = e^{i\pi(2n+1)}, \text{ where } n \in \mathbb{Z}$$

$n = [-\infty, \infty]$

$$z_{0n} = \frac{i\tau\pi(2n+1)}{2} = i\tau\pi(n+1/2)$$

In the complex plane:



take contour in upper half

let $R \to \infty \Rightarrow n = [0, 1, \dots, \infty]$

$n \in \mathbb{Z}$

$E(\omega) \equiv 2E_0 I$, where $I = \oint_c f(z)\, dz = 2\pi i \sum$ residues

residues $\equiv a_{-1n} = \lim_{z \to z_{0n}} \{(z - z_{0n})\, f(z)\} = \lim_{z \to i(n+1/2)\pi\tau} \{(z - z_{0n}) \dfrac{q(z)}{g(z)}$

$$a_{-1n} = e^{i(\omega-\omega_0)[i\pi\tau(n+1/2)]} \lim_{z\to z_{0n}} \left\{ \frac{(z-z_{0n})}{g(z)} \right\}$$

$$g(z) = \underbrace{\cancel{g(z_{0n})}}_{\text{zero by defn.}} + \left.\frac{dg}{dz}\right|_{z_{0n}} (z-z_{0n}) + \frac{1}{2}\frac{d^2g}{dz^2}(z-z_{0n})^2 + \cdots$$

$$\therefore \lim_{z\to z_{0n}} = \frac{1}{dg/dz|_{z_{0n}}} = \left(\frac{dg}{dz}\right)^{-1}_{z_{0n}}$$

$$\therefore a_{-1n} = \exp\{-(\omega-\omega_0)^2\pi\tau(n+1/2)\} \left(\frac{dg}{dz}\right)^{-1}_{z_{0n}}$$

$$\left.\frac{d}{dz}g(z)\right|_{z_{0n}} = \frac{1}{\tau}\left\{e^{z/\tau} - e^{-z/\tau}\right\}_{z_{0n}} = \frac{1}{\tau}\left\{\exp[i\pi(n+1/2)] - \exp[-i\pi(n+1/2)]\right\}$$

$$= \frac{i2}{\tau}\sin[\pi(n+1/2)] = \frac{i2}{\tau}(-1)^n$$

$$\Rightarrow a_{-1n} = \frac{\tau}{i2}(-1)^n e^{-(\omega-\omega_0)\pi\tau/2} e^{-\alpha n}, \quad \text{where } \alpha = (\omega-\omega_0)\pi\tau$$

$$I = 2\pi i \sum_{n=0}^{\infty}\left(\frac{\tau}{i2}(-1)^n e^{-\alpha n} e^{-\alpha/2}\right) = \pi\tau e^{-\alpha/2}\sum_{n=0}^{\infty}\left(-e^{-\alpha}\right)^n = \frac{\pi\tau}{e^{\alpha/2}+e^{-\alpha/2}}$$

$$\therefore I = \frac{\pi\tau}{2}\operatorname{sech}\left(\frac{\pi\tau}{2}[\omega-\omega_0]\right)$$

$$\boxed{E(\omega) = \pi\tau E_0 \operatorname{sech}\left(\frac{\pi\tau}{2}[\omega-\omega_0]\right)}$$

Q.2 $\operatorname{sech}^2(x) = 1/2 \Rightarrow x = \cosh^{-1}(\sqrt{2}) = 0.88$ $\qquad (\Omega = \omega - \omega_0)$

Time: $x = t/\tau \Rightarrow \tau_p = 2(x\tau) = 1.76\tau$, Freq: $x = \pi\tau/2\,\Omega \Rightarrow \Delta\omega_p = 2\left(\frac{2x}{\pi\tau}\right) = \frac{1.122}{\tau}$

$\therefore$ TBP: $\tau_p\Delta\omega_p = 1.122(1.76) = 1.97 \equiv 2\pi C_B \Rightarrow \boxed{C_B = 0.315}$

$$\Omega = \omega - \omega_0$$

Q.3 Note: $|E(t)|^2/|E(t)|^2 \propto I(t)/I(t) \Rightarrow$ ignore $E(t) \to I(t)$ factors in conversion

also $\mathrm{sech}(t)$ = even func, $\int \text{odd} \times \text{even} = 0 \Rightarrow \langle \tau_1 \rangle = 0$

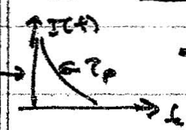
$$\langle \tau_2 \rangle^2 = \frac{\int_{-\infty}^{\infty} t^2 \,\mathrm{sech}^2(t/\tau)\, dt}{\int_{-\infty}^{\infty} \mathrm{sech}^2(t/\tau)\, dt} = \frac{\pi^2 \tau^2}{12} \Rightarrow \boxed{\begin{aligned} \langle \tau \rangle = \tau_{RMS} = \frac{\pi}{\sqrt{12}}\tau \\ \tau_{RMS} \sim 0.907\,\tau \end{aligned}}$$

b.1

$$E(\omega) = E_0 \int_0^{\infty} e^{-t/\tau} e^{-i(\omega-\omega_0)t}\, dt = E_0 \int_0^{\infty} e^{-[\tau^{-1} + i\Omega]t}\, dt = \left. \frac{e^{-[\tau^{-1}+i\Omega]t}}{-[\tau^{-1}+i\Omega]} \right|_0^{\infty}$$

$$\therefore \boxed{E(\omega) = E_0 [\tau^{-1} + i\Omega]^{-1}}$$

b.2



Time: $e^{-2\tau_p/\tau} = 1/2 \Rightarrow \tau_p = \frac{1}{2} \ln 2\, \tau$

Freq: $\frac{|E(\Omega)|^2}{|E(0)|^2} = \frac{\tau^2 [1 + (\tau\Omega)^2]^{-1}}{\tau^2}$

$\therefore \Omega_{1/2} \Rightarrow 1 + (\tau \Omega_{1/2})^2 = 2$

$\Omega_{1/2} = 1/\tau, \quad \Delta\omega_p = 2/\tau$

TBP: $\tau_p \Delta\omega_p = \frac{1}{2} \ln 2\, \tau \left(\frac{2}{\tau}\right) = \ln(2) = 2\pi C_B \Rightarrow \boxed{C_B = \frac{\ln(2)}{2\pi}}$

$C_B \sim 0.1103$

b.3 $U \propto E_0^2 \int_0^{\infty} e^{-2t/\tau}\, dt = -\frac{\tau}{2} e^{-2t/\tau} \Big|_0^{\infty} = \frac{\tau}{2} E_0^2$

$$\langle \tau_2 \rangle^2 = \left[\frac{E_0^2}{U} \int_0^{\infty} t^2 e^{-2t/\tau}\, dt - \left(\frac{E_0^2}{U} \int_0^{\infty} t\, e^{-2t/\tau}\, dt \right)^2 \right]$$

$$= \left[\frac{2}{\tau} \left\{ -\frac{\tau}{2} e^{-2t/\tau} \left(t^2 + t\tau + \frac{\tau^2}{2} \right) \right\}_0^{\infty} - \left(\frac{2}{\tau} \left\{ -\frac{\tau}{2} e^{-2t/\tau} \left(t + \tau/2 \right) \right\}_0^{\infty} \right)^2 \right]$$

$$= \left[\frac{\tau^2}{2} - \frac{\tau^2}{4} \right] = \frac{\tau^2}{4} \Rightarrow \boxed{\langle \tau_2 \rangle = \tau/2}$$

(2) No: If the pulse duration is kept constant and a chirp is added, then additional bandwidth is required to support the pulse duration. An LSI system can not add new bandwidth.

It IS possible to add chirp to an LSI system if $\Delta\omega_p$ is kept fixed. This happens when the pulse propagates through a dispersive LSI system.