1 The Driving term in Maxwell's propagation equation

1.1 Index of refraction due to electron

The index of refraction seems a trivial subject that we had been introduced to since our high school studies through a Snell's law. In the middle of twentieth century there were a few attempts to have a better understanding of the index of refraction. With a classical mechanics picture in mind the derivation of index was a combination of the principle of least action and Fermat's principle. (Reference:The Refractive Index in Electron Optics and the Principles of Dynamics 1949 Proc. Phys. Soc. B 62 8, http://iopscience.iop.org/0370-1301/62/1/303). Often the index of refraction is only associated with bond electrons and treated as it was introduced in Chapter IV of "theory of electrons" by Lorentz.[1]

In textbooks like Klein and Furtak (chapter one, or attached hw1.pdf, part (b)), the index of refraction is calculated by solving the damped harmonic oscillator of electron motion with applied electric field "E". The position and momentum are calculated. At this point one approach is taking the position "Δr" of the electron motion and claim that the dipole is simply "$e\Delta r$ and by multiplying the dipole with the density of the electrons the polarization "$P = \chi\epsilon_0 E = Ne\Delta r$" is known and the index is simply $n = \sqrt{1+\chi}$ Another approach, generally applied to plasma, is looking at the current density (from the velocity of the electron) and inserting "Nev" as "J" in Maxwells' equation

$$\nabla \times H = J_f + \frac{\partial D}{\partial t} = J + \epsilon_0 \frac{\partial E}{\partial t}. \tag{1}$$

In the absence of magnetization and free current $M = 0$ and $J_f = 0$, the velocity of the electron "v" gives the current density "J". Here is a unspoken trick , *the "ϵ" is considered a property of the medium and constant over the changes of the electromagnetic field. There are situations in physics where this does not apply anymore, when ϵ itself is a function of time and changes over the applied light signal.*

1.2 The bound electron

The classical approach is to calculate the motion of the bound electron, modeled as a dipole. The electron is at a (small) distance d from the positive ion. It oscillates with the applied electric field. This is the classical oscillator model. The Coulomb field produces a restoring force, which defines a resonance frequency. One introduces a damping term. A similar model is used for the plasma. The result is that, away from resonance, under the influence of an optical oscillating field at ω, the motion of the electron follows the frequency of the applied field, in phase, and is thus $d = d_0 \cos\omega t$. At a point of observation at a distance R from the dipole, the field due to the dipole is:

$$\Delta E = \frac{q^2}{4\pi R^2}\left[1 - \frac{R^2}{(R+d)^2}\right] \approx \frac{2q^2 d}{4\pi R^3} \tag{2}$$

Putting that in Maxwell's propagation equation:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 (E + \Delta E)}{\partial t^2} = 0 \tag{3}$$

[1] An interesting review of the index with application to mixed gases can be found in "The Refractive Index of an Ionized Medium. Author(s): C. G. Darwin, Reviewed work(s); Source: Proceedings of the Royal Society of London., Vol. 146, No. 856 (Aug. 1, 1934), pp. 17-46 Published by: The Royal Society URL: http://www.jstor.org/stable/2935475 .Accessed: 06

or

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \Delta E}{\partial t^2} = \frac{\omega^2}{c^2}\Delta E. \tag{4}$$

Using:

$$\begin{aligned} E &= \frac{1}{2}\mathcal{E}e^{i(\omega t - kz)} \\ \Delta E &= \frac{1}{2}\Delta E a e^{i(\omega t - kz)} \end{aligned}$$

we find:

$$-2ik\frac{\partial \mathcal{E}}{\partial z} - 2i\frac{\omega}{c^2}\frac{\partial \mathcal{E}}{\partial t} = \frac{\omega^2}{c^2}\Delta\mathcal{E}, \tag{5}$$

and

$$\frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = -i\frac{\omega}{2c}\Delta\mathcal{E}, \tag{6}$$

We started with $\Delta\mathcal{E}$ in phase with the applied field. After insertion in the propagation equation, it appears that its envelope is adding 90 degrees out of phase with the applied field, as is the case of an index of refraction.

1.3 Other approach: source polarization in Maxwell's equation

For a propagating wave along z, the propagation of any function $f(z - ct)$ would be solution of a first order differential equation:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) f = 0. \tag{7}$$

[which amount to $f'_z - (1/c)f'_t = 0$.] Maxwell's wave equation is a bit different: it could be seen as the product of two wave equations along $+z$ and $-z$: (right propagation)$\times$(left propagation)

$$\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t} \times \frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{8}$$

The right hand side of Maxwell's equations refers to local sources – not propagating ones, hence no $\partial/\partial z$ associated with them. Magnetic field effects are neglected in the derivation of plasma frequencies, polarization, approximation incompatible with propagation. Indeed, the magnetic field is an integrant part of Maxwell's propagation equation.

Recalling the derivation of Maxwell's propagation equation :

$$\begin{aligned} \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \times B &= \mu_0 J + \mu_0 \frac{\partial(\epsilon_0 E + P)}{\partial t} \\ \nabla \times \nabla \times E &= -\frac{\partial(\nabla \times B}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \mu_0\frac{\partial^2 P}{\partial t^2} \\ & \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0\frac{\partial^2 P}{\partial t^2} \end{aligned} \tag{9}$$

Another way to look at it: instead of P, we are adding the local electric field ΔE from the radiation of the electron. Either of:

1. say $\Delta E = P/\epsilon_0$ hence
$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \Delta E}{\partial t^2}$$
,

2. The local electric field adds to the applied field only in the time derivative:
$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 (E + \Delta E)}{\partial t^2} = 0$$

ΔE may be in phase with the applied field, but, as a function of z, and in the retarded frame of reference, $\Delta\tilde{\mathcal{E}}$ is $\pi/2$ out of phase with $\partial\tilde{\mathcal{E}}/\partial z$. The standard propagation equation for the envelope
$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} = -i\frac{\Omega}{c}\Delta\tilde{\mathcal{E}} \tag{10}$$
comes from making the substitution:
$$\begin{aligned} E &= \frac{1}{2}\tilde{\mathcal{E}}e^{i(\omega t - kz)} \\ \Delta\tilde{\mathcal{E}} &= \frac{1}{2}\Delta\tilde{\mathcal{E}}e^{i(\omega t - kz)} \end{aligned}$$

The left side of Maxwell's propagation equation becomes zero to first order by making $\omega^2/c^2 = k^2$, while the right hand side becomes either $\mu_0\omega^2 P$ or $\omega^2/c^2\Delta\tilde{\mathcal{E}}$.
$$\begin{aligned} \left(-2ik\frac{\partial}{\partial z} - 2\frac{\omega}{c^2}i\frac{\partial}{\partial t}\right)\tilde{\mathcal{E}} &= -\frac{\omega^2}{c^2}\Delta\tilde{\mathcal{E}} \\ \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\tilde{\mathcal{E}} &= -\frac{i\omega}{2c}\Delta\tilde{\mathcal{E}}, \end{aligned}$$

or, going to the retarded frame $t \leftarrow t - z/c$ and $z \leftarrow z$:
$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} = \frac{i\omega}{2c}\Delta\tilde{\mathcal{E}}. \tag{11}$$

It appears as if by the time the electron re-radiates, the wave has already moved by a distance of $\lambda/4$.