The Fabry-Perot Cavity

Contents

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1 Basics

We will consider for simplicity a symmetric Fabry-Perot cavity. The boundaries of the Fabry-Perot are air (outside, medium 1) glass (inside, medium 2) interfaces. We will use the following notations:

- $\tilde{t}_{12}$ = transmission from outside (1) to inside (2)
- $\tilde{t}_{21}$ = transmission from inside (2) to outside (1)
- $\tilde{r}_{12}$ = reflection from outside (1) to inside (2)
- $\tilde{r}_{21}$ = reflection from inside (2) to outside (1).

The incident field is a plane wave of amplitude unity.

1.1 Field transmission

$$\begin{aligned}\mathcal{T} &= \tilde{t}_{12}\tilde{t}_{21}e^{-ikd} \\ &\quad+\tilde{t}_{12}\tilde{t}_{21}\left(e^{-2ikd}\cdot\tilde{r}_{21}\tilde{r}_{21}\right)e^{-ikd} \\ &\quad+\tilde{t}_{12}\tilde{t}_{21}e^{-ikd}\left(e^{-2ikd}\cdot\tilde{r}_{21}\tilde{r}_{21}\right)^2+\dots \\ &= \tilde{t}_{12}\tilde{t}_{21}e^{-ikd}\frac{1}{1-\tilde{r}_{21}^2e^{-2ikd}}.\end{aligned} \tag{1}$$

1.1.1 Interface properties

For an asymmetric interface:

$$\boxed{\tilde{t}_{12}\tilde{t}_{21}-\tilde{r}_{12}\tilde{r}_{21}=1} \tag{2}$$

and

$$\boxed{\tilde{r}_{12}=-\tilde{r}_{21}^*} \tag{3}$$

Equation (2) implies that we can do the following substitution in Eq. (1:

$$\tilde{t}_{12}\tilde{t}_{21}=1+\tilde{r}_{12}\tilde{r}_{21}=1-|r_{12}|^2=1-R. \tag{4}$$

The result for the field transmission is:

$$\boxed{\mathcal{T}(\Omega)=\frac{(1-R)e^{-ikd}}{1-Re^{i\delta}}} \tag{5}$$

where

$$\boxed{\delta(\Omega)=2\varphi_r-2k(\Omega)d} \tag{6}$$

1.2 Field reflection

$$\boxed{\mathcal{R}(\Omega) = \frac{\sqrt{R}\left(e^{i\delta} - 1\right)}{1 - Re^{i\delta}}.} \tag{7}$$

One can easily verify that, if — and only if — kd is real:

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1 \tag{8}$$

Equations (5) and (7) are the transfer functions for the Fourier transform of the field. The dependence on the frequency argument Ω occurs through $k = n(\Omega)\Omega/c$.

1.3 Examples

Transmission for a train of pulses.
Fabry-Perot as a frequency filter.

1.4 Transfer functions

A transfer function is the mathematical representation of the relation between the input and output of a system.

$\mathcal{R}(\Omega)$, $\mathcal{T}(\Omega)$ are examples of transfer functions for the field $\tilde{\mathcal{E}}(\Omega)$.

2 Frequency Filter

Referring to Eq. (5), $\delta = -2k(\Omega).d$ for normal incidence. For clarity, we will neglect the phase shift on reflection.

One can use either angular or cyclic frequencies:

$$\delta = -\frac{2nd}{c}\Omega \tag{9}$$

$$\delta = -\frac{4\pi nd}{c}\nu \tag{10}$$

There are two important parts in the transmission: its periodicity, i.e. the transmission takes the same value for increments of δ by $2N\pi$. This is called the *free spectral range.* In angular frequencies:

$$\boxed{\Delta\Omega_{\text{fsr}} = \frac{\pi c}{nd}} \tag{11}$$

In cyclic frequencies:

$$\boxed{\Delta\nu_{\text{fsr}} = \frac{c}{2nd}} \tag{12}$$

The next important dependence is close to the peak transmission, which corresponds to $\delta = 0$ or $2N\pi$. The best approach is to make the approximation of small δ in the trigonometric function.

WARNING One cannot make the approximation of δ small in $\tilde{\mathcal{T}}$ and thereafter calculate $T = |\tilde{\mathcal{T}}|^2$. One has to FIRST calculate T, and THEREAFTER make the approximation of small δ. The intensity transmission factor T is:

$$T = \frac{1}{1 + \frac{2R}{(1-R)^2}\cos\delta}. \tag{13}$$

The approximation $\cos\delta \approx 1 - \delta^2/2$ in Eq. (13) yields:

$$T \approx \frac{1}{1 + \frac{R}{(1-R)^2}\delta^2}. \tag{14}$$

Making the approximation of small δ in Eq. 5 first gives you a different result. The FWHM of this Lorentzian is:

$$\Delta\delta_{\text{res}} = \frac{2(1-R)}{\sqrt{R}}. \tag{15}$$

This relation leads directly to the definition of the finesse, which is the ratio of the free spectral range (2π) to the linewidth:

$$F = \frac{\pi\sqrt{R}}{1-R}. \tag{16}$$

For the use of the Fabry-Perot as a Filter, one can define the FWHM in angular frequency

$$\boxed{\begin{aligned} \Delta\Omega_{\rm res} &= \frac{c(1-R)}{nd\sqrt{R}} \quad (17) \\ \Delta\nu_{\rm res} &= \frac{c(1-R)}{2\pi nd\sqrt{R}} \quad (18) \end{aligned}}$$

or cyclic frequency: (18)

2.1 Fabry-Perot Cascade

Let us assume that the thinnest practical Fabry-Perot to be of 100 μm thickness. The corresponding free spectral range is $\Delta\nu_{\rm fs1} = 1.5 \cdot 10^{12}$ Hz. The bandwidth that needs to be filtered is often much larger. Let us assume a "square" bandwidth that covers exactly $(2N+1)\Delta\nu_{\rm fs1}$ We dan arrange to have the transmission peak of index zero just outside the band, the transmission peak of index 1 just inside, the transmission peak of index $N+1$ in the middle, index $2N+1$ just inside and index $2N+1$ just outside.

We want to built a filter that leaves only one peak transmitted in that range. Let us use a Fabry Perot of approximately twice the thickness, and the same finesse. "Approximately twice the thickness" implies a free spectral range of

$$\Delta\nu_{\rm fs2} = \Delta\nu_{\rm fs1}\left(\frac{1-\epsilon}{2}\right). \tag{19}$$

Since the finesse is the same:

$$\Delta\nu_{\rm res2} \approx \frac{1}{2}\Delta\nu_{\rm res1}. \tag{20}$$

We look now for conditions that will tell us what the minimum $\Delta\nu_{\rm res1}$ should be in order to have only the central peak surviving in the superposition of the two Fabry-Perots. The first condition is that the second peak from the center of the thicker FP does not overlap with the first peak away from the center of the thinner one:

$$\begin{aligned} \Delta\nu_{\rm fs1} - 2\Delta\nu_{\rm fs2} &\geq \Delta\nu_{\rm res1} \\ \epsilon \times \Delta\nu_{\rm fs1} &\geq \Delta\nu_{\rm res1} \end{aligned} \tag{21}$$

The second condition is that the outer transmission peaks do not overlap:

$$\begin{aligned} (2N+1)\Delta\nu_{\rm fs2} - N\Delta\nu_{\rm fs1} &\geq \Delta\nu_{\rm res1} \\ \left[\frac{1}{2} - (N+\frac{1}{2})\epsilon\right]\Delta\nu_{\rm fs1} &\geq \Delta\nu_{\rm res1} \end{aligned} \tag{22}$$

which can be satisfied if $\epsilon < 1/(2N+1)$.

3 Transmission/reflection for a monochromatic Gaussian beam

For a Gaussian beam:

$$\mathcal{E} = \mathcal{E}_0 e^{-r^2/w^2} \tag{23}$$

The Fourier transform along the transverse dimension is:

$$\mathcal{E}(\Delta k) = \int_{-\infty}^{\infty} \mathcal{E}(r) e^{i\Delta k r} dr \propto e^{-(\Delta k)^2 w^2/4}. \tag{24}$$

In the Fabry-Perot transmission function, we write $\vec{k} = \vec{k_0} + \vec{\Delta k}$, with the vector $\vec{\Delta k}$ orthogonal to the vector $\vec{k_0}$. In the Fabry-Perot transmission function:

$$\delta = 2\varphi_r - 2\vec{k}_0.\vec{d} - 2\vec{\Delta k}.\vec{d} = 2\varphi_r - 2k_0 d\cos\theta + 2\Delta k d \sin\theta = \delta_0 + 2a\Delta k, \tag{25}$$

with $a = d\sin\theta$ To first order, we can neglect the variation of θ as compared to θ, putting all the variation in Δk.

The transmission of a Gaussian beam to a Fabry-Perot — in $k - space$ is:

$$e^{-(\Delta k)^2 w^2/4} \times \frac{(1-R)e^{-i(\vec{k_0}.\vec{d}+2a\Delta k)}}{1 - Re^{i(\delta_0 + 2a\Delta k)}} \tag{26}$$

One can get a lot of information from this expression, without having to make the inverse Fourier transform to the position space. The phase factor $\vec{k_0}.\vec{d} + 2a\Delta k$ disappears when on takes the absolute value square. The important phase factor is in the denominator: $\delta_0 + 2a\Delta k$. Δk varies essentially in the range $\pm 1/w$.

Near normal incidence

Depending on the exact angle of incidence, δ_0 can be close to 0 or π. The term $2a\Delta k = 2\Delta k d \sin\theta$ varies between $\pm 2(d/w)\sin\theta$. Starting with no fringe, there will be one fringe across the beam if $2(d/w)\sin\theta \geq 2\pi$, i.e. narrow beam ($w$ small) and/or long FP (d large).

4 Fabry-Perot – simplest case – geometric model

We will consider that the phase shift δ is real, and is only due to the propagation factor: $\delta = -kd - \phi$. The intensity transmission of a Fabry-Perot is given by:

$$\begin{aligned} T &= \frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2\frac{\phi}{2}} \\ &= \frac{1}{1 + \left(\frac{2F}{\pi}\right)^2 \sin^2\frac{\phi}{2}} \\ &\approx \frac{1}{1 + \frac{R}{(1-R)^2}\phi^2}. \end{aligned} \tag{27}$$

where the approximation is valid for $(1-R) \ll 1$, and $\phi \ll 1$. Within that approximation, the transmission function is a Lorentzian of width (FWHM) $\Delta\phi_t$:

$$\Delta\phi_t = 2\frac{\sqrt{R}}{(1-R)}. \tag{28}$$

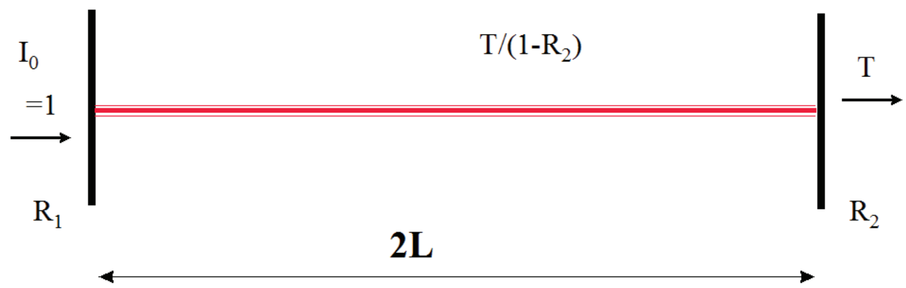
In the above expressions, $R = \sqrt{R_1 R_2}$, R_1 and R_2 being the (intensity) reflection coefficients of the mirrors, and $\phi = k(2L) = 4\pi nL/\lambda = 4\pi\nu nL/c$ is the round-trip phase shift. The free spectral range, or the frequency spacing between transmission peaks, is $\Delta\nu = c/(2nL)$. The finesse F is the ratio of the full width half maximum of each transmission peak $\Delta\nu_t$ to the free spectral range $\Delta\nu_{ax} = c/(2nL)$.

$$F = \frac{\pi\sqrt{R}}{1-R} \tag{29}$$

Another form for the Fabry-Perot transmission is:

$$T = \frac{(1-R)^2}{1+R^2-2R\cos\phi} = \left[\frac{\epsilon}{\mathcal{D}}\right]^2. \tag{30}$$

The transmission of the Fabry-Perot can be determined by a simple geometrical construction, as sketched in Fig. 1(b). The incident field is represented by a unit vector OA with a phase angle ϕ. The quantity $\mathcal{D} = \sqrt{1-2R\cos\phi+R^2}$ is the side AB of the triangle, where B is the point at $\epsilon = 1-R$ from the intersection of the circle with the $\phi = 0$ axis. It is easily seen from the figure that the transmission factor $[\epsilon/\mathcal{D}]^2$ varies between unity and $[(1-R)/(1+R)]^2$.



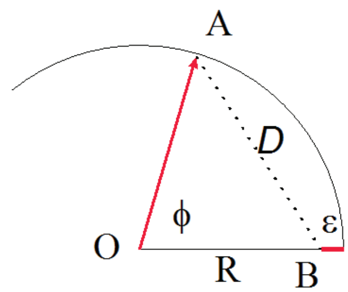


Figure 1: Fabry-Perot cavity (a) and geometrical representation of its transmission factor (b).

5 Fabry-Perot with gain

Phase factor δ: $\tilde{\delta} = \delta - ia$,

$$\boxed{\delta(\Omega) = 2\varphi_r - 2k(\Omega)d} \tag{31}$$

The result for the field transmission is:

$$\boxed{\mathcal{T}(\Omega) = \frac{(1-R)e^{a/2}e^{-ikd}}{1-Re^{a}e^{i\delta}}} \tag{32}$$

At resonance (δ and $kd = 0$):

$$\mathcal{T}(\Omega) = \frac{(1-R)e^{a/2}}{1-Re^{a}}$$

Substituting into the expressions for the transmission [Eq. (5)] and reflection [Eq. (7)] and taking the absolute value squared to find the intensity transmission factor:

$$\begin{aligned} |\mathcal{R}|^2 &= \frac{(1-R)^2}{1+Re^{2a}-2Re^{a}\cos\delta} \\ |\mathcal{T}|^2 &= \frac{R\left[e^{2a}+1-2e^{a}\cos\delta\right]}{1+Re^{2a}-2Re^{a}\cos\delta}. \end{aligned} \tag{33}$$

The traditional approach to simplify these expression is to make the substitution $\cos\delta = 1 - 2\sin^2(\delta/2)$. The result is:

$$\boxed{\begin{aligned} |\mathcal{R}|^2 &= \frac{R\left(e^a-1\right)^2}{\left(1-Re^a\right)^2} \frac{\left[1+\frac{4e^a}{(e^a-1)^2}\sin^2\frac{\delta}{2}\right]}{1+\frac{4Re^a}{(1-Re^a)^2}\sin^2\frac{\delta}{2}} && (34) \\ |\mathcal{T}|^2 &= \frac{(1-R)^2}{\left(1-Re^a\right)^2}\frac{1}{1+\frac{4Re^a}{(1-Re^a)^2}\sin^2\frac{\delta}{2}} && (35) \end{aligned}}$$

For values of δ close to the resonance condition ($N\pi$), the shape of the transmission function is a Lorentzian, of half width determined by the condition:

$$\frac{4Re^a}{(1-Re^a)^2}\sin^2\frac{\delta}{2} = 1. \tag{36}$$

As the gain a increases, the transmission at resonance increases, to reach infinity at threshold ($Re^a = 1$). The linewidth decreases to zero at the threshold gain.

The reflection at $\delta = 0$ is:

$$|\mathcal{R}|^2 = \frac{R\left(1-e^a\right)^2}{(1-Re^a)^2} \tag{37}$$

For small a, this expression can be approximated as:

$$\frac{a^2 R}{(1-Re^a)^2} \tag{38}$$

which also tends to infinity at threshold.

Let us sent through this Fabry-Perot a probe beam, at a frequency tuned to the half width (i.e. transmission of the intensity = 0.5) of the empty (no gain) Fabry-Perot. The detuning for the empty cavity is given by the condition:

$$\sin^2\frac{\delta}{2} = \frac{(1-R)^2}{4R}. \tag{39}$$

If this value is substituted in the expression for the transmission, we find:

$$|\mathcal{T}|^2 = x\left[\frac{1}{1+xe^a}\right] \tag{40}$$

The transmission is the limit for $x \to \infty$ of this expression, where

$$x = \frac{(1-R)^2}{(1-Re^a)^2}. \tag{41}$$

The solution is

$$\frac{x}{1+xe^a} \to e^{-a} \approx 1. \tag{42}$$

Thus, even though the linewidth tends to zero and the transmission to infinity, there is still a transmission factor of unity at the optical frequency corresponding to 50% transmission for the empty cavity.

The intensity inside the resonator is simply the transmitted intensity divided by the transmission coefficient of the second mirror:

$$I_i = \frac{I_0}{|\mathcal{T}|^2} \frac{1}{1-R}. \tag{43}$$

The threshold condition $Re^a = 1$ corresponds to the onset of laser oscillation. In the considerations above, we have assumed the gain a to be a constant, independent of the intensity. In reality, the gain a is itself a function of I_i:

$$a = \frac{a_0}{1 + \frac{I_i}{I_s}}, \tag{44}$$

where a_0 is the unsaturated absorption coefficient, and I_s is the saturation intensity.

Given an unsaturated gain above threshold, and an input intensity I_0, the intensity will build up inside the cavity, and the gain will decrease, until reaching the equilibrium condition that corresponds to the gain = loss:

$$Re^{a_0/(1+I_i/I_s)} = 1 \tag{45}$$

which is the threshold condition defined previously through Eqs. (34) and (35).

5.1 Absorbing Fabry-Perot

The absorbing Fabry-Perot equivalent to the Fabry-Perot with gain, except that the gain factor a is negative.

6 Impulse response of a Fabry-Perot

The transmission of any signal — cw, pulse, train of pulses — through a Fabry-Perot can be calculated by taking the product of the Fourier transform of the pulse and the transfer function Eq. (5). The result of this operation is the Fourier transform of the transmitted pulse. For instance, cw radiation corresponds in the frequency domain to a $\delta(\Omega - \omega_\ell)$ function. The transmitted field through the Fabry-Perot is therefore simply the value of the function (5) at the frequency ω_ℓ $(\times \mathcal{E}_0)$. For a very short pulse that covers N modes of the Fabry-Perot, the Fourier transform of the transmission is a comb of N Fabry-Perot transmission peaks. The Fourier transform of that frequency comb is a pulse train. Because the envelope of these peaks has the same shape as the Fourier transform of the incident pulse, the pulses of the sequence have the same shape and duration as the incident pulse. The shape of the envelope of the pulse sequence is the inverse Fourier transform of a Fabry-Perot transmission peak. Since the latter is a Lorentzian, the pulse train follow a decaying exponential. The latter situation is easier to analyze in the time domain. A single pulse, shorter than the mirror spacing, is sent to the Fabry-Perot. There are no interferences. The pulse rattles inside the cavity, losing a fraction $(1 - R)$ at each reflection.

We consider next the more complex problem of the transmission/reflection of a Fabry-Perot to a train of pulses. Here again, the numerical approach would be to multiply the Fourier spectrum of the pulse train by the transfer function Eq. (5) for transmission, or Eq. (7) for the reflection. The direct derivation in time however gives a better understanding of the resonator response.

We consider the irradiation of a Fabry-Perot by the output of a mode-locked laser of which the cavity is 4 $\times$ longer than the Fabry-Perot. Each pulse of the train has a duration much smaller than the round-trip time of the Fabry-Perot.

Let us first derive an expression for the electric field of the transmitted pulse(s) through this Fabry-Perot irradiated by the mode-locked laser. We will make step by step derivation, following the pulse as it enters the resonator and travels back and forth in that cavity, keeping in mind the ratio of cavity and pulse rates.

The field transmission factor is given by:

$$\begin{aligned} \mathcal{T} &= \tilde{t}_{12} e^{-ikd} \tilde{t}_{21} + \tilde{t}_{12} \tilde{r}_{21}^{8} e^{-9ikd} \tilde{t}_{21} + \tilde{t}_{12} \tilde{r}_{21}^{16} e^{-19ikd} \tilde{t}_{21} \ldots \\ &= \frac{\tilde{t}_{12} \tilde{t}_{21} e^{-ikd}}{1 - \tilde{r}_{21}^{8} e^{-i8\delta}} \end{aligned} \tag{46}$$

For the intensity transmission:

$$|\mathcal{T}|^2 = \frac{(1-R)^2}{1 + R^8 - 2R^4 \cos 8\delta}. \tag{47}$$

If we make the standard replacement $\cos 8\delta = 1 - 2\sin^2 4\delta$, the equation for the transmitted intensity becomes:

$$|\mathcal{T}|^2 = \frac{(1-R)^2}{1 + R^8 - 2R^4(1 - 2\sin^2 4\delta}$$

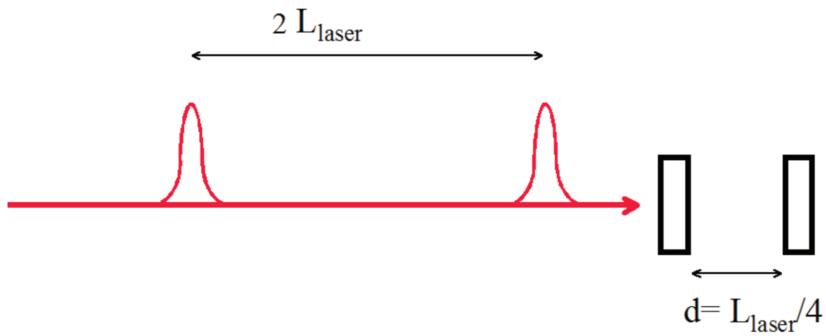


Figure 2: Fabry-Perot cavity irradiated by a mode-locked pulse train of 4 times lower repetition rate than the round-trip rate of that Fabry-Perot.

$$
\begin{aligned}
&= \frac{1}{(1+R+R^2+R^3)^2 + \frac{4R^4}{(1-R)^2}\sin^2 4\delta} \\
&= \frac{1}{(1+R+R^2+R^3)^2}\frac{1}{1+\frac{4R^4}{(1-R)^2(1+R+R^2+R^3)^2}\sin^2 4\delta}
\end{aligned} \tag{48}
$$

This is to be compared to the transmission of the Fabry-Perot for cw light:

$$
\left[|\mathcal{T}|^2\right]_{\mathrm{cw}} = \frac{1}{1+\frac{4R}{(1-R)^2}\sin^2\frac{\delta}{2}}. \tag{49}
$$

In the case of the cw FP, this represents near the resonance (for R close to unity) a Lorentzian of full width at half maximum (FWHM):

$$
\Delta\delta_{cw} = \frac{1-R}{\sqrt{R}} = \frac{T}{\sqrt{R}}. \tag{50}
$$

For this case of mode-locked input, the FWHM of the Lorentzian is:

$$
\Delta\delta_{ml} = \frac{(1-R)(1+R+R^2+R^3)}{8R^2} \approx \frac{T}{2R^2}. \tag{51}
$$

where the approximation $R \approx 1$ was made for the sum in the numerator. One could argue that the Fabry-Perot is 4 times longer as seen by the mode-locked train, therefore the

FWHM should be 4 times narrower. On the other hand, the losses due to the reflections not synchronized with the feeding pulse contribute to a broadening of the transmission peak (term in R^2 instead of $\sqrt{R}$ in Eq. (51).

The Fabry-Perot transmission at resonance is the expression Eq. (47) for $\delta = 0$:

$$\begin{aligned} |\mathcal{T}|^2 &= \frac{(1-R)^2}{(1-R^4)^2} \\ &= \left(\frac{1}{1+R+R^2+R^3}\right)^2 \approx \frac{1}{16}, \end{aligned} \tag{52}$$

where the approximation applies to a high Q cavity ($R \approx 1$).

6.0.1 Reflected and transmitted energy in the limit $R \to 1$

Let W_1 be the energy of each incident pulse. Since there are four transmitted pulses for each incident pulse, the total transmitted energy is $4W_1/16 = W_1/4$, where W_1 is the energy of incident light. The reflected energy should therefore be $(3W_1/4)$, which comprises 3 pulses with energy $W/16$ and one pulse with energy $W_1 \times [(12/16) - (3/16)] = 9W_1/16$, in order to satisfy energy conservation. The reflected field of $3\mathcal{E}_1/4$ results from destructive interference of the incident field (amplitude $\mathcal{E}_1$) and the transmitted field from the Fabry-Perot (amplitude $\mathcal{E}_1/4$).

7 The nonlinear Fabry-Perot

7.1 Saturable gain of absorption off resonance

As we have seen, the laser can be seen as a nonlinear Fabry-Perot, where the linewidth and tranmission factor are dependent on the gain, itself dependent on the intracavity intensity, itself dependent on the transmission factor. If radiation is injected in such a Fabry-Perot with gain off resonance, the index of refraction is also dependent on the intensity, hence on the transmission, which is also dependent through δ on the index... One can feel that this problem can become rather complex.

7.2 Nonlinear index of refraction

We will consider here the simplest nonlinearity, that of the Kerr effect, or an index of refraction linearly dependent on the intensity:

$$n = n_0 + n_2 I_i \tag{53}$$

where the subscript i of the intensity indicates that we are concerned with the intracavity intensity.

7.3 A Fabry-Perot cavity as an ultrafast switch

Any change in cavity length, or change in index, will shift the resonance. Since the resonance can be very narrow, one would see here an ideal opportunity to make an ultrafast switch. Let us assume for simplicity that the two mirrors are identical, and that there is no phase shift on reflection. The phase shift is then $\delta = 2\varphi_r - 2kd = 2\varphi_r + \phi = \phi$. For the transmitted intensity T given above, the intensity inside the device is $T/(1-R)$. If the Fabry-Perot is filled with a medium with a nonlinear index ($n = n_0 + n_2 I$), the phase shift ϕ is a function of the transmission:

$$\phi = \phi_0 + \frac{4\pi n_2 T I_0 L}{\lambda(1-R)} = \phi_0 + \frac{T I_0}{a}. \tag{54}$$

Equation (54) can be represented by straight lines on a graph of T versus ϕ:

$$T = \frac{a}{I_0}\left(\phi - \phi_0\right). \tag{55}$$

Starting from the situation sketched in Fig. 3, as the incident intensity increases from an initial value of 0 [vertical line for Eq. (55)] the transmission become multivalued. The Fabry-Perot has become a multistable element. The intersections of the straight lines in Fig. 3 with the Fabry-Perot transmission curves are made to construct the transmission characteristic of the bistable Fabry-Perot of Fig. 4.

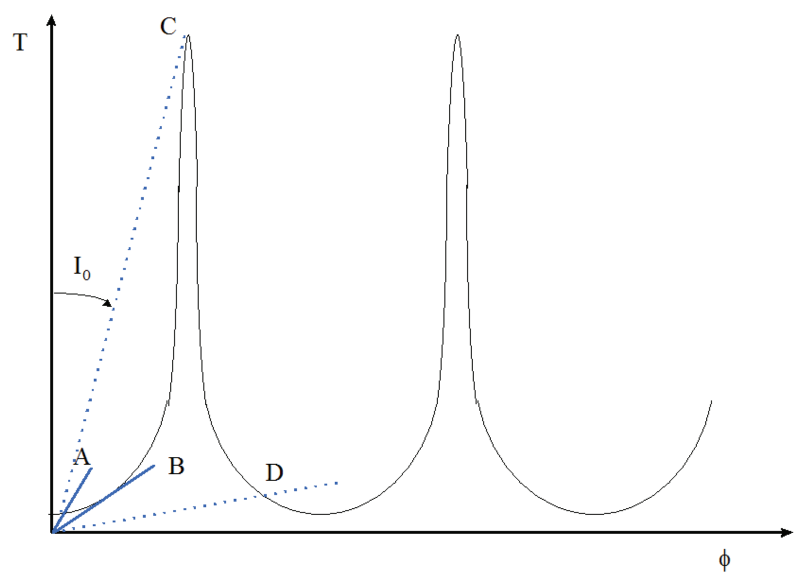


Figure 3: Fabry-Perot transmission versus the phase factor ϕ (a), which has become itself a linear function of transmission (straight lines of decreasing slope as the intensity I_0 is increasing).

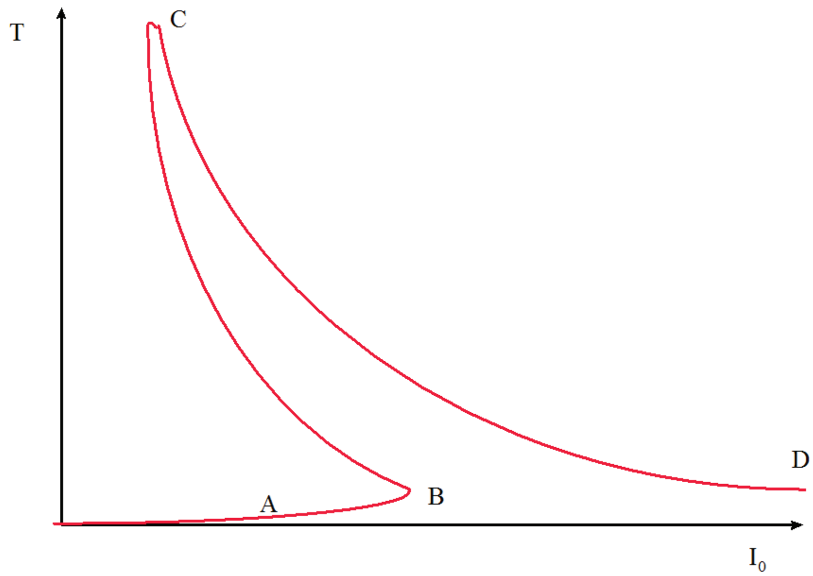


Figure 4: Fabry-Perot transmission versus the input intensity, constructed graphically from the previous figure. Corresponding points A, B, C and D are labeled on both figures.

7.4 Hidden approximation in the previous approach

The result of the previous section is that the output of the Fabry-Perot is multivalued, and may switch from one value to another as the input is being varied. There is however an inconsistency in that approach: we are assuming the input constant, but then make conclusions as to the changes in output as the input is varied! The graphical derivation of the nonlinear Fabry-Perot made in the previous section implicitly assumes that the spectrum of the field is a $\delta-$ function. In other words, the variations of the field is assumed to be slow compared to even the lifetime $\tau_c = \tau_{RT}/(1-R)$ of the Fabry-Perot.

When the Fourier transform of the input electric field is not a $\delta-$ function, we have to use the correct general expression for the transfer function of a Fabry-Perot:

$$\boxed{\mathcal{T}(\Omega) = \frac{(1-R)e^{-ikd}}{1-Re^{i\delta}}} \tag{56}$$

which implies that the transmitted field is:

$$\boxed{\tilde{\mathcal{E}}_{out} = \mathcal{T}(\Omega)\tilde{\mathcal{E}}_{in}} \tag{57}$$

where $\tilde{\mathcal{E}}_{in}$ is the field incident from the left on the Fabry-Perot, and $\tilde{\mathcal{E}}_{out}$ is the transmitted field to the right. Equation (57) takes into account all the dynamics of the field and of the Fabry-Perot.

In the case of the nonlinear Fabry-Perot, the phase factor $\delta = 2\varphi_r + \phi$ itself is a function of the field $\tilde{\mathcal{E}}(\Omega)$ *inside* the Fabry-Perot. As in the previous section, we will assume that the phase shift upon reflection φ_r at the interfaces is zero, and $\delta = \phi$. Instead of Eq.(54), we have to write:

$$\begin{aligned}
\phi &= \phi_0 + \Delta\delta \\
&= \phi_0 + \frac{2\Omega n_2 |\tilde{\mathcal{E}}(\Omega)|^2}{\pi\sqrt{\frac{\mu_0}{\epsilon}}} \\
&= \phi_0 + \frac{2\Omega n_2 |\mathcal{T}(\Omega)\tilde{\mathcal{E}}_{in}(\Omega)|^2}{(1-R)\pi\sqrt{\frac{\mu_0}{\epsilon}}} \\
\phi &= \phi_0 + a\Omega|\mathcal{T}(\Omega)\tilde{\mathcal{E}}_{in}(\Omega)|^2
\end{aligned} \tag{58}$$

To solve the problem of the nonlinear Fabry-Perot, we have so substitute Eq. (58) into Eq. (56). The resulting equation can be solved for $\mathcal{T}$ for each value of Ω. A numerical algorithm to solve the self-consistent Eq. (56) would be by successive iterations. As a first iteration, one could take the zero-field solution for $\mathcal{T}_0(\Omega)$, insert this solution in Eq. (58) to find the second iteration $\mathcal{T}_1(\Omega)$, *etc* ...

7.5 Possible approximations

The most interesting situation is when $R \leq 1$ is close to unity; the Fabry-Perot resonance is narrow, and the input pulse has a spectral with of the order of the Fabry-Perot resonance width. We will consider — even though this might be a coarse approximation — that the phase shift δ is sufficiently small to make the approximation $\sin\delta \approx \delta$. For simplicity also, let us assume that the phase shifts upon reflection can be neglected, and that the nonlinear phase shift is small, hence $kd \approx \delta/2 \approx \phi_0/2$. Therefore, the transfer function of the Fabry-Perot, in the frequency range of interest, can be approximated as follows:

$$\begin{aligned}
\mathcal{T}(\Omega) &\approx \frac{i(1-R)e^{i(\phi_0+\pi)/2}}{1-R(1+i\phi_0+i\Delta\delta)} \\
&= \frac{i(1-R)e^{i(\phi_0+\pi)/2}}{1-R-i\phi_0-i\Delta\delta} \\
&\approx \frac{-(1-R)e^{i(\phi_0+\pi)/2}}{\phi_0+\Delta\delta(\Omega)} \\
&= \frac{-(1-R)e^{i(\phi_0+\pi)/2}}{\phi_0+a\Omega|\mathcal{T}(\Omega)\tilde{\mathcal{E}}_{in}(\Omega)|^2}
\end{aligned} \tag{59}$$

In this equation, for each value of Ω, one can solve Eq. (59) to find the transmission function $y = \mathcal{T}(\Omega)$. The equation to solve for each value of Ω is:

$$\phi_0 y + a\Omega\tilde{\mathcal{E}}_{in}|y|^2 y + (1-R)\phi_0/2 = 0. \tag{60}$$

If the input field is a symmetrical function in time, $\tilde{\mathcal{E}}_{in}$ is a real function, and it is clear from Eq. (60) that the imaginary part of the solution is zero. In this particular condition, Eq. (60) can be written:

$$\phi_0 y + a\Omega\tilde{\mathcal{E}}_{in} y^3 + (1-R)\phi_0/2 = 0. \tag{61}$$

Once $\mathcal{T}(\Omega)$ has been determined, the transmitted signal can be calculated by taking the inverse Fourier Transform of $\tilde{\mathcal{E}}_{out} = \mathcal{T}(\Omega)\tilde{\mathcal{E}}_{in}$.

7.6 Other example of nonlinear Fabry-Perot

It is not only a nonlinear index that can provide a nonlinearity. A medium with saturable gain makes the Fabry-Perot even more nonlinear, because of the gain narrowing with intensity.

For instance, the medium inside the Fabry-Perot can be an homogeneously broadened gain medium. The real and imaginary parts of the gain are:

$$\alpha = \frac{\alpha_0}{1+\Delta\omega^2 T_2^2 + \frac{I}{I_s}} \tag{62}$$

$$\alpha_i = \frac{\alpha_0 \Delta\omega T_2}{1+\Delta\omega^2 T_2^2 + \frac{I}{I_s}} \tag{63}$$

where $\Delta\omega = \omega_0 - \Omega$.

The transmission function of the Fabry-Perot is:

$$\mathcal{T}(\Omega) = \frac{(1-R)e^{-ikd}}{1-Re^{i\delta}} \tag{64}$$

with

$$\delta = \delta_0 - \alpha_i - i\alpha. \tag{65}$$

In the transmission function, we can make the approximation that $\delta \ll 1$.

$$\mathcal{T}(\Omega) = \frac{1}{1-\frac{R(i\alpha+\alpha_i)}{1-R}} \approx 1 + \frac{R}{1-R}\left[\alpha_0 \frac{1+i(\omega_0-\Omega)T_2}{1+\Delta\omega^2 T_2^2 + \frac{I}{I_s}}\right] \tag{66}$$

The intracavity intensity is:

$$|\tilde{\mathcal{E}}(\Omega)|^2 = \frac{|\tilde{\mathcal{E}}_0(\Omega)\mathcal{T}(\Omega)|^2}{1-R}. \tag{67}$$

8 Fabry-Perot with moving end mirror

A laser cavity is generally a Fabry-Perot, operating at a resonance peak. If a mirror is displaced by only a fraction of wavelength, the resonance peak moves. Thus all the light should escape the Fabry-Perot, and the laser emission has to restart from noise. This brings up the question: how can the emission of any laser — for instance a He-Ne laser — be stable? Anytime a mirror position moves because of vibration or thermal expansion, all the light should escape from the cavity?

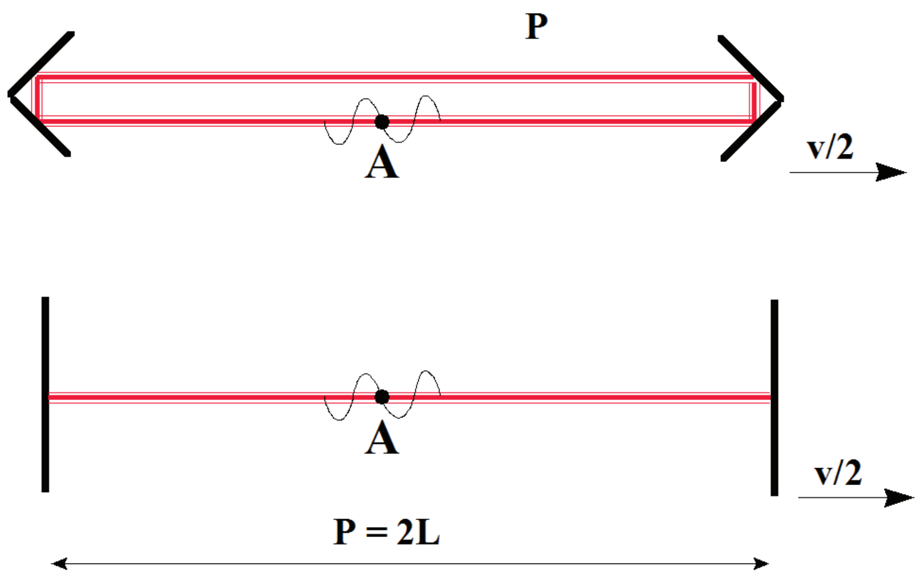


Figure 5: At a point A of a ring cavity (top figure), any wave having made a round-trip should constructively interfere with itself in order for the cavity to be resonant. The same argument applies to a simple linear Fabry-Perot cavity (lower part of the figure). Can the resonance condition be maintained if the end of the cavity moves with a velocity $v/2$?

Let us consider either a ring laser — the elongated rectangular cavity, sketched in the upper Fig. 5, of perimeter P, or a linear cavity of length $L = P/2$ (lower part of the figure).

The right reflector of the ring cavity is a corner cube moving away at a velocity $v/2$, and the right mirror of the linear cavity is moving at the same velocity.

The resonance condition of that cavity is that the wave makes constructive interference with itself at any point A. This does not imply a standing wave in the ring cavity, but simply a round-trip condition:

$$kP = \frac{\omega_0}{c} P_0 = 2N_i \pi \qquad \text{or} \qquad \omega_0 = \frac{2N_i \pi}{P_0} c, \tag{68}$$

where N_i is the mode index (integer). It is assumed that the index of refraction of the medium is unity (or we simply would have to replace c by c/n)

Adiabatic motion of mirror M Mirror M (on the right) moves away at a velocity $v/2$. That means that the cavity perimeter is now $P(t) = P_0 + vt$. The cavity mode drifts according to:

$$\omega(t) = \frac{2N_i\pi c}{P_0(1 + vt/P_0)} \approx \omega_0(1 - \frac{v}{P_0}t). \tag{69}$$

In one cavity round-trip time $\tau_{RT} = P_0/c$, the frequency shift is:

$$\omega_0 \frac{v}{P_0}\tau_{RT} = \omega_0\frac{v}{c}. \tag{70}$$

However, *at each reflection on the mirror*, there is a Doppler shift $\omega_D = \Omega_0 v/c$. Thus, the frequency change by Doppler shift keeps the cavity resonant, and therefore slow fluctuations in the cavity length do not result in light escaping from the cavity. Slow fluctuations in cavity length therefore result in a frequency noise, and no amplitude noise of a laser cavity. Note that the cavity length change can equivalently be made by modulating the index of refraction in the cavity (for instance with a electro-optic modulator).

The basic assumption made above is not that the mirror M moves "slowly", but that the mirror velocity be constant over a round-trip time. This is essentially an acceleration condition. The change in Doppler shift over one round-trip should be small compared with the width of the resonance, which is $\Delta\nu = 1/t_c$ (where t_c is the lifetime of the cavity):

$$\begin{aligned} \Delta(\omega_D) &= \omega_0\frac{v(t+\tau_{RT}) - v(t)}{c} \\ &= \frac{\omega_0\tau_{RT}}{c}\frac{dv}{dt} \ll \frac{2\pi}{t_c} = 2\pi\frac{c}{P_0 F} = \frac{2c(1-R)}{P_0\sqrt{R}}. \end{aligned} \tag{71}$$

The second line of Eq. (71) gives the most physical expression for the maximum acceleration:

$$\frac{dv}{dt} \ll \frac{\lambda}{t_c\tau_{RT}}. \tag{72}$$

The mirror should not accelerate to a velocity of a wavelength/(cavity round-trip-time) in less than a cavity lifetime. Substituting the value of the cavity lifetime:

$$\frac{dv}{dt} \ll \frac{1-R}{\pi\sqrt{R}}\frac{\lambda}{\tau_{RT}^2}. \tag{73}$$

9 Fabry-Perot with gain, laser and injection

Let us come back to the general expression for a Fabry-Perot transmission:

$$\boxed{\mathcal{T}(\Omega) = \frac{(1-R)e^{-ikd}}{1-Re^{i\delta}}} \tag{74}$$

where

$$\boxed{\delta(\Omega) = 2\varphi_r - 2k(\Omega)d} \tag{75}$$

A laser is a Fabry-Perot with gain; complex in general, and function of frequency. We have defined: $\tilde{\delta} = \delta - ia$, where $\tilde{\delta}$ is the previously defined complex phase factor, δ is the real part of this phase factor, and $a = a(\Omega)$. The gain factor/round-trip is $\mathcal{G}(\Omega) = \exp[a(\Omega)]$. Let us assume for now that we are close to resonance, and that the frequency dependence of the gain is negligible near the Fabry-Perot resonance. The expression (74) for the Fabry-Perot with gain becomes:

$$\mathcal{T}(\Omega) = \frac{(1-R)e^{-ikd}}{1 - R\mathcal{G}(\Omega)\cos\delta - iR\mathcal{G}(\Omega)\cos\delta} \tag{76}$$

Close to resonance, and assuming for simplicity that the phase shift on reflection is zero, $\delta = (\Omega - \Omega_0)P/c = (\Omega - \Omega_0)\tau_{RT}$ is small; cos can be replaced by unity, and sin by the angle. In addition, $R \approx 1$. Thus, the expression for the gain reduces to:

$$\frac{E_{out}}{E_{in}} = \tilde{g}(\Omega) = \frac{1-R}{1 - \mathcal{G} + i\mathcal{G}(\Omega - \Omega_0)\tau_{RT}}. \tag{77}$$

The gain clamps at unity, for the free running oscillation. However, there is still gain for an external signal applied near resonance, as sketched in Fig. 6. The intensity amplification for the signal injected near resonance is: The intensity amplification for the injected signal is:

$$|\tilde{g}(\Omega)|^2 \approx \frac{(1-R)^2}{[(\Omega - \Omega_0)\tau_{RT}]^2} = \frac{1}{[(\Omega - \Omega_0)\tau_c]^2}, \tag{78}$$

where $\tau_c = \tau_{RT}/(1-R)$ is the lifetime of the cavity.

The free-running oscillation has an output intensity I_0. The externally applied signal has an intensity I_1 at a frequency ω_1, and produces an output intensity $|\tilde{g}(\omega_1)|^2 I_1$, independently of the output of the oscillator at ω_0. If that intensity exceeds I_0, the externally applied signal will saturate the gain, and there will only be an oscillating signal at the frequency ω_1. Lock-in occurs if the amplified injected signal exceeds the free-running oscillation:

$$|\tilde{g}(\omega_1)|^2 I_1 = \frac{I_1}{[(\Omega - \Omega_0)\tau_c]^2} \geq I_0 \tag{79}$$

which leads to the detuning range in which the external source of intensity I_1 is sufficient to "lock" the frequency of the oscillator to ω_1:

$$\omega - \omega_0 < \frac{1}{\tau_c}\sqrt{\frac{I_{inj}}{I_0}}. \tag{80}$$

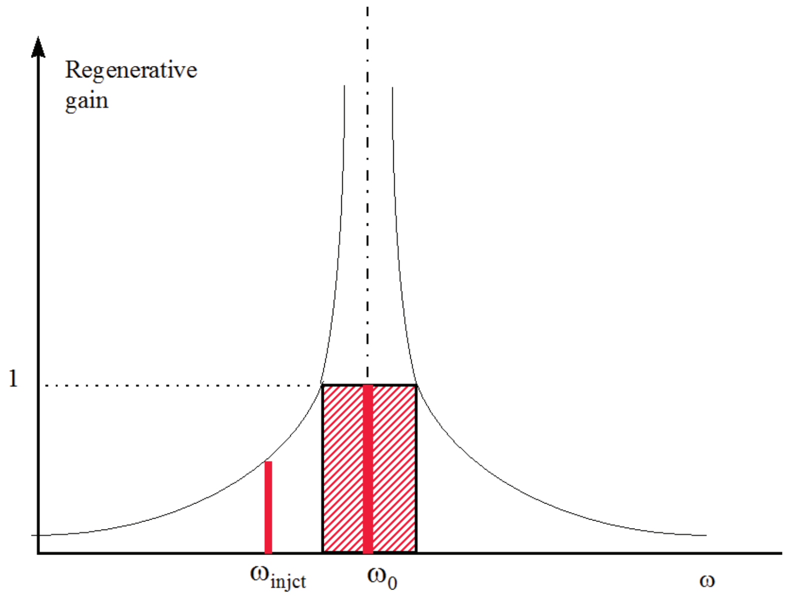


Figure 6: Regenerative gain near the cavity resonance. The hatched region indicates lock-in.

This is for an *externally* applied signal. In the case of a laser gyro, two signals of different frequency circulate inside the resonator in opposite sense of circulation. There is frequency difference $\Delta\omega$ due to rotation of the cavity around an axis perpendicular to the cavity between these two intracavity beams. If Ω_R is the rotation rate of the laser, $\Delta\omega = R\Omega_R$, where $R = 4A/\lambda P$ (P = laser perimeter, A its area and λ the wavelength). Scattering at intracavity element (for instance the mirrors) couples one beam (for instance the clockwise circulating) into the other (the counterclockwise). If the laser has an output power I_0, the internal intensity is $I_0/(1-R)$. If r is the scattering coefficient per round-trip, a fraction r or the internal intensity in one direction is injected into the other. That internal injection of $rI_0/(1-R)$ corresponds to an external injection $rI_0/(1-R)^2$, which corresponds to I_1 above. We find the condition for lock-in by substitution in Eq. (79):

$$|\tilde{g}(\omega_1)|^2 I_1 = |\tilde{g}(\omega_1)|^2 rI_0/(1-R)^2 = \frac{rI_0}{[(\Omega-\Omega_0)\tau_{RT}]^2} \geq I_0. \tag{81}$$

Lock-in in a laser gyro occurs thus for $\Delta\omega \approx r/\tau_{RT}$.