The Fabry-Perot Cavity

We will consider for simplicity a symmetric Fabry-Perot cavity. The boundaries of the Fabry-Perot are air (outside, medium 1) glass (inside, medium 2) interfaces. We will use the following notations:

- $\tilde{t}_{12}$ = transmission from outside (1) to inside (2)
- $\tilde{t}_{21}$ = transmission from inside (2) to outside (1)
- $\tilde{r}_{12}$ = reflection from outside (1) to inside (2)
- $\tilde{r}_{21}$ = reflection from inside (2) to outside (1).

The incident field is a plane wave of amplitude unity.

Field transmission

$$\begin{aligned} \mathcal{T} &= \tilde{t}_{12}\tilde{t}_{21}e^{-ikd} \\ &\quad + \tilde{t}_{12}\tilde{t}_{21}\left(e^{-2ikd}\cdot\tilde{r}_{21}\tilde{r}_{21}\right)e^{-ikd} \\ &\quad + \tilde{t}_{12}\tilde{t}_{21}e^{-ikd}\left(e^{-2ikd}\cdot\tilde{r}_{21}\tilde{r}_{21}\right)^2 + \ldots \\ &= \tilde{t}_{12}\tilde{t}_{21}e^{-ikd}\frac{1}{1-\tilde{r}_{21}^2 e^{-2ikd}}. \end{aligned} \tag{1}$$

Interface properties

For an asymmetric interface:

$$\boxed{\tilde{t}_{12}\tilde{t}_{21} - \tilde{r}_{12}\tilde{r}_{21} = 1} \tag{2}$$

and

$$\boxed{\tilde{r}_{12} = -\tilde{r}_{21}^*} \tag{3}$$

Equation (2) implies that we can do the following substitution in Eq. (1:

$$\tilde{t}_{12}\tilde{t}_{21} = 1 + \tilde{r}_{12}\tilde{r}_{21} = 1 - |r_{12}|^2 = 1 - R. \tag{4}$$

The result for the field transmission is:

$$\boxed{\mathcal{T}(\Omega) = \frac{(1-R)e^{-ikd}}{1-Re^{i\delta}}} \tag{5}$$

where

$$\boxed{\delta(\Omega) = 2\varphi_r - 2k(\Omega)d} \tag{6}$$

Field reflection

$$\boxed{\mathcal{R}(\Omega) = \frac{\sqrt{R}\left(e^{i\delta} - 1\right)}{1 - Re^{i\delta}}}. \tag{7}$$

One can easily verify that, if — and only if — kd is real:

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1 \tag{8}$$

Equations (5) and (7) are the transfer functions for the Fourier transform of the field. The dependence on the frequency argument Ω occurs through $k = n(\Omega)\Omega/c$.

Examples

Transmission for a train of pulses.
Fabry-Perot as a frequency filter.

Transfer functions

A transfer function is the mathematical representation of the relation between the input and output of a system.

$\mathcal{R}(\Omega)$, $\mathcal{T}(\Omega)$ are examples of transfer functions for the field $\tilde{\mathcal{E}}(\Omega)$.

Frequency Filter

Referring to Eq. (5), $\delta = -2k(\Omega).d$ for normal incidence. For clarity, we will neglect the phase shift on reflection.

One can use either angular or cyclic frequencies:

$$\delta = -\frac{2nd}{c}\Omega \tag{9}$$

$$\delta = -\frac{4\pi nd}{c}\nu \tag{10}$$

There are two important parts in the transmission: its periodicity, i.e. the transmission takes the same value for increments of δ by $2N\pi$. This is called the *free spectral range.* In angular frequencies:

$$\boxed{\Delta\Omega_{\mathrm{fsr}} = \frac{\pi c}{nd}} \tag{11}$$

In cyclic frequencies:

$$\boxed{\Delta\nu_{\mathrm{fsr}} = \frac{c}{2nd}} \tag{12}$$

The next important dependence is close to the peak transmission, which corresponds to $\delta = 0$ or $2N\pi$. The best approach is to make the approximation of small δ in the trigonometric function.

WARNING One cannot make the approximation of δ small in $\tilde{\mathcal{T}}$ and thereafter calculate $T = |\tilde{\mathcal{T}}|^2$. One has to FIRST calculate T, and THEREAFTER make the approximation of small δ. The intensity transmission factor T is:

$$T = \frac{1}{1 + \frac{2R}{(1-R)^2}\cos\delta}. \tag{13}$$

The approximation $\cos\delta \approx 1 - \delta^2/2$ in Eq. (13) yields:

$$T \approx \frac{1}{1 + \frac{R}{(1-R)^2}\delta^2}. \tag{14}$$

Making the approximation of small δ in Eq. 5 first gives you a different result. The FWHM of this Lorentzian is:

$$\Delta\delta_{\mathrm{res}} = \frac{2(1-R)}{\sqrt{R}}. \tag{15}$$

This relation leads directly to the definition of the finesse, which is the ratio of the free spectral range (2π) to the linewidth:

$$F = \frac{\pi\sqrt{R}}{1-R}. \tag{16}$$

For the use of the Fabry-Perot as a Filter, one can define the FWHM in angular frequency

$$\boxed{\begin{array}{lcll} \Delta\Omega_{\mathrm{res}} & = & \dfrac{c(1-R)}{nd\sqrt{R}} & (17) \\ \Delta\nu_{\mathrm{res}} & = & \dfrac{c(1-R)}{2\pi nd\sqrt{R}} & (18) \end{array}}$$

or cyclic frequency: (18)

Fabry-Perot Cascade

Let us assume that the thinnest practical Fabry-Perot to be of 100 μm thickness. The corresponding free spectral range is $\Delta\nu_{\mathrm{fs1}} = 1.5 \cdot 10^{12}$ Hz. The bandwidth that needs to be filtered is often much larger. Let us assume a "square" bandwidth that covers exactly $(2N+1)\Delta\nu_{\mathrm{fs1}}$ We dan arrange to have the transmission peak of index zero just outside the band, the transmission peak of index 1 just inside, the transmission peak of index $N+1$ in the middle, index $2N+1$ just inside and index $2N+1$ just outside.

We want to built a filter that leaves only one peak transmitted in that range. Let us use a Fabry Perot of approximately twice the thickness, and the same finesse. "Approximately twice the thickness" implies a free spectral range of

$$\Delta\nu_{\mathrm{fs2}} = \Delta\nu_{\mathrm{fs1}} \left(\frac{1-\epsilon}{2} \right). \tag{19}$$

Since the finesse is the same:

$$\Delta\nu_{\mathrm{res2}} \approx \frac{1}{2} \Delta\nu_{\mathrm{res1}}. \tag{20}$$

We look now for conditions that will tell us what the minimum $\Delta\nu_{\mathrm{res1}}$ should be in order to have only the central peak surviving in the superposition of the two Fabry-Perots. The first condition is that the second peak from the center of the thicker FP does not overlap with the first peak away from the center of the thinner one:

$$\begin{aligned} \Delta\nu_{\mathrm{fs1}} - 2\Delta\nu_{\mathrm{fs2}} &\geq \Delta\nu_{\mathrm{res1}} \\ \epsilon \times \Delta\nu_{\mathrm{fs1}} &\geq \Delta\nu_{\mathrm{res1}} \end{aligned} \tag{21}$$

The second condition is that the outer transmission peaks do not overlap:

$$\begin{aligned} (2N+1)\Delta\nu_{\mathrm{fs2}} - N\Delta\nu_{\mathrm{fs1}} &\geq \Delta\nu_{\mathrm{res1}} \\ \left[\frac{1}{2} - (N + \frac{1}{2})\epsilon \right] \Delta\nu_{\mathrm{fs1}} &\geq \Delta\nu_{\mathrm{res1}} \end{aligned} \tag{22}$$

which can be satisfied if $\epsilon < 1/(2N+1)$.

Transmission/reflection for a monochromatic Gaussian beam

For a Gaussian beam:

$$\mathcal{E} = \mathcal{E}_0 e^{-r^2/w^2} \tag{23}$$

The Fourier transform along the transverse dimension is:

$$\mathcal{E}(\Delta k) = \int_{-\infty}^{\infty} \mathcal{E}(r) e^{i\Delta k r} dr \propto e^{-(\Delta k)^2 w^2/4}. \tag{24}$$

In the Fabry-Perot transmission function, we write $\vec{k} = \vec{k_0} + \vec{\Delta k}$, with the vector $\vec{\Delta k}$ orthogonal to the vector $\vec{k_0}$. In the Fabry-Perot transmission function:

$$\delta = 2\varphi_r - 2\vec{k}_0.\vec{d} - 2\vec{\Delta k}.\vec{d} = 2\varphi_r - 2k_0 d \cos\theta + 2\Delta k d \sin\theta = \delta_0 + 2a\Delta k, \tag{25}$$

with $a = d \sin\theta$ To first order, we can neglect the variation of θ as compared to Δk, putting all the variation in Δk.

The transmission of a Gaussian beam to a Fabry-Perot — in $k - space$ is:

$$e^{-(\Delta k)^2 w^2/4} \times \frac{(1-R)e^{-i(\vec{k_0}.\vec{d}+2a\Delta k)}}{1 - Re^{i(\delta_0 + 2a\Delta k)}} \tag{26}$$

One can get a lot of information from this expression, without having to make the inverse Fourier transform to the position space. The phase factor $\vec{k_0}.\vec{d} + 2a\Delta k$ disappears when on takes the absolute value square. The important phase factor is in the denominator: $\delta_0 + 2a\Delta k$. Δk varies essentially in the range $\pm 1/w$.

Near normal incidence

Depending on the exact angle of incidence, δ_0 can be close to 0 or π. The term $2a\Delta k = 2\Delta k d \sin\theta$ varies between $\pm 2(d/w)\sin\theta$. Starting with no fringe, there will be one fringe across the beam if $2(d/w)\sin\theta \geq 2\pi$, i.e. narrow beam ($w$ small) and/or long FP (d large).