3

Optical Fibers with Single-Step Index Profiles

The single-step index profile is the simplest fiber design that employs homogeneous dielectrics, and it thus provides a logical starting point in the study of fibers of more complicated design. This basic profile was seen in most of the early manufactured fibers and is still found in many that are currently made. Although more sophisticated profile designs have largely replaced the single-step, the analysis of these is often simplified by referring to an "equivalent" step index fiber, whose light-guiding properties are similar. This association is possible, since the single-step profile analysis yields qualitative mode characteristics and mode designations that apply to fibers of any profile in which the refractive index maximizes on the fiber axis.

The single-step index fiber (hereafter referred to as step index) consists of two concentric homogeneous dielectrics, as shown in Fig. 3.1. The inner dielectric, or core, fills the region $r < a$ and is of refractive index n_1. The outer material ($r > a$), known as the cladding, is of index n_2, where $n_2 < n_1$. The cladding is considered, for purposes of analysis, to be of infinite outer radius. Key struc-

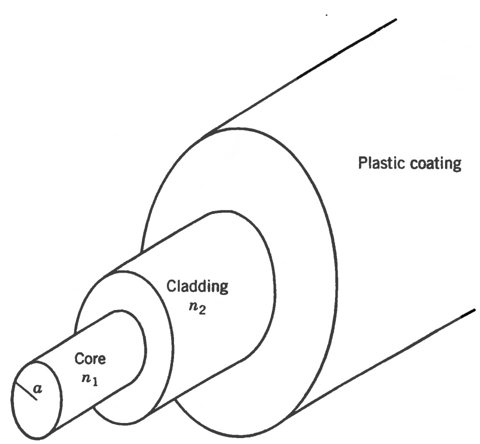


Figure 3.1. Step index fiber design.

tural parameters include the core radius, a, and the normalized index differenc (usually expressed in percent):

$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \approx \frac{n_1 - n_2}{n_1} = \frac{\delta n}{n_1} \qquad (3.1$$

The approximation in (3.1) arises from assuming that n_1 is very close in value tc n_2, which is almost always true. Another important quantity is the *numericai aperture*, *N.A.*, defined as the sine of the maximum angle (measured outside the fiber with respect to the z axis) of an incident light ray that becomes totally confined within the fiber. By using simple geometrical arguments (left as an exercise) it can be shown that this maximum angle is achieved when the interior ray propagates at the critical angle, measured at the core–cladding interface. By using this fact, along with the relation $\sin\theta_c = n_2/n_1$, it is found that

$$N.A. = \sqrt{n_1^2 - n_2^2} = n_1\sqrt{2\Delta} \qquad (3.2)$$

Single-mode and multimode step index fibers form two classifications of practical concern. Standards on dimensions have been set by various committees. Among these are the Electronic Industries Association (EIA), the International Electrotechnical Commission (IEC), and the Comité Consultatif International Télégraphique et Téléphonique (CCITT). All have specified a cladding diameter of 125 μm as the production standard for most single-mode and multimode fibers. Core diameters vary between 50 and 100 μm for multimode, and between 4 and 10 μm for single-mode fibers. Values of Δ are typically about 0.2% for single mode and about 1% for multimode.

As mentioned, the step index profile has in many instances been replaced by more complicated profiles for special purposes. In particular, multimode fibers with graded index profiles exhibit significantly increased bandwidth over the step index case. Alternative profiles for single-mode fibers have become important for minimizing dispersion over specified wavelength ranges and for control of mode polarization. Some of these special designs will be considered in Chapter 6. Their mode field characteristics and frequency behavior are qualitatively similar to those found in the step index profile, and the mode designation system is the same.

3.1. RAY PROPAGATION IN THE STEP INDEX FIBER

In studying the slab waveguide, it was found that ray propagation studies enabled one to derive the eigenvalue equations through transverse resonance, in addition to being able to obtain a physical pict tions. In the slab and fiber, the precise mode fields can in fact be constructed through an appropriate superposition of plane waves (characterized by rays). This procedure becomes complicated, however, when dealing with core sizes on the order of a wavelength. In the present treatment, a simple ray picture will be used to obtain qualitative features of the modes that will be recognized and used later when solving the wave equation.

Cylindrical coordinates are used, with the fiber axis coincident with the z axis. Radius r, the azimuthal angle ϕ, and z are the three coordinates; their respective directions are specified by unit vectors, $\hat{\mathbf{a}}_r$, $\hat{\mathbf{a}}_\phi$, and $\hat{\mathbf{a}}_z$. As in the slab guide, a guided mode in a fiber can be constructed from a group of rays that totally reflect from the interface at $r = a$. Ray propagation in a fiber is complicated by the possibility of a path component in the ϕ direction, thus giving rise to a *skew* ray. Such a ray exhibits a spiral-like path down the core, never crossing the z axis. A *meridional* ray is one that has no ϕ component; it passes through the z axis and is thus in direct analogy to a slab guide ray.

Figure 3.2 shows a skew ray at radius r in the core, along with its components in the three coordinate directions. The ray has propagation constant magnitude $k_1 = n_1 k_0$. This value is related to the component magnitudes by

$$\beta_r^2 + \beta_\phi^2 + \beta^2 = n_1^2 k_0^2 \tag{3.3}$$

The net transverse propagation constant is written as $\beta_t^2 = \beta_r^2 + \beta_\phi^2$. The ϕ component, providing a measure of the skewness, is restricted by the requirement that when multiplied by the core circumference at radius r, it yields a net phase

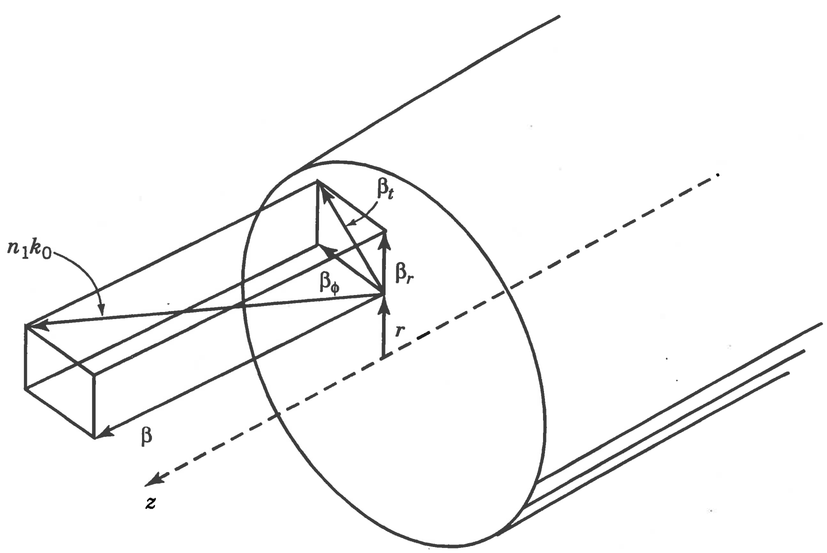


Figure 3.2. Skew ray decomposition in the core of a step index fiber.

shift of $2q\pi$, where q is an integer. Its value will therefore be $\beta_\phi = q/r$. The component, β, is the propagation constant of the fiber mode that is associat with the ray.

In analogy to the slab waveguide, a fiber guided mode is established wh its plane wave components totally reflect at the core–cladding boundary, a when ray trajectories are such that transverse resonance is established with the core. A ray propagates through a single round trip as it moves from a giv starting point to a point at an increased distance in z at which it again achiev its starting orientation. Transverse resonance is established when the startir and finishing points of the ray path, having the same r and ϕ coordinates, ha the same relative phase.

The condition for total reflection is established by considering the ray path it reflects from a curved interface, as discussed in Section 2.7. A meridional ra will have no component in the direction of curvature (ϕ direction) and will th experience total reflection when its incidence angle exceeds the critical angl A skew ray, on the other hand, has a propagation component in ϕ, characterize by propagation constant $\beta_\phi = q/r$. At a critical radius within the cladding, th ϕ-directed phase velocity will thus exceed that of light in the cladding, and s a mode with only a ϕ component would radiate away. It is the z componer of the ray that allows mode confinement. The net propagation constant for confined wave will by definition have no radial component in the cladding. Th wavevector magnitude in the cladding will therefore be $k_2 = (\beta_\phi^2 + \beta^2)^{1/2}$. A infinite distances, β_ϕ approaches zero, which means that β must be greater tha $n_2 k_0$ to prevent the mode from radiating. The restriction on β is thus identica to that of the slab guide; that is, $n_2 k_0 < \beta < n_1 k_0$.

In the case of meridional rays, it is possible to have TE and TM field con figurations. Skew rays, on the other hand, cannot have pure TE or TM polar izations. This is because the slanted plane wave will change its orientation o each reflection, making it impossible to maintain purely transverse electric o magnetic fields over the entire ray path. The mode fields associated with a skew ray will thus have z components of both **E** and **H**. For this reason, these modes are known as "hybrid" modes and are given the designation EH or HE.

3.2. FIELD ANALYSIS OF THE STEP INDEX FIBER

Field analysis begins by assuming field solutions of the form

$$\mathbf{E} = \mathbf{E}_0(r,\phi)\exp(-j\beta z) \qquad \mathbf{H} = \mathbf{H}_0(r,\phi)\exp(-j\beta z) \tag{3.4}$$

As in the slab guide, the z component of **E** is again found using Eq. (2.25), rewritten in cylindrical coordinates:

$$\nabla_t^2 E_{z1} + (n_1^2 k_0^2 - \beta^2)E_{z1} = 0 \qquad r < a$$

$$\nabla_t^2 E_{z2} + (n_2^2 k_0^2 - \beta^2) E_{z2} = 0 \qquad r \geq a \tag{3.6}$$

where $(n_1^2 k_0^2 - \beta^2) = \beta_{t1}^2$ and $(n_2^2 k_0^2 - \beta^2) = \beta_{t2}^2$. Assuming transverse variation in both r and ϕ, the wave equation becomes, in general,

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \beta_t^2 E_z = 0 \tag{3.7}$$

It is assumed that the solution for E_z will be a discrete series of modes, each of which has separated dependences on r, ϕ, and z in product form:

$$E_z = \sum_i R_i(r)\Phi_i(\phi)\exp(-j\beta_i z) \tag{3.8}$$

Each term (mode) in the expansion must itself be a solution of (3.7). A single mode, $E_z = R\Phi \exp(-j\beta z)$, can thus be substituted into (3.7) to obtain

$$\frac{r^2}{R}\frac{d^2 R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + r^2\beta_t^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \tag{3.9}$$

The left-hand side of (3.9) depends only on r, whereas the right-hand side depends only on ϕ. Since r and ϕ vary independently, it must follow that each side of (3.9) must be equal to a constant. Defining this constant as q^2, (3.9) separates into the following two equations:

$$\frac{d^2\Phi}{d\phi^2} + q^2\Phi = 0 \tag{3.10}$$

and

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\beta_t^2 - \frac{q^2}{r^2}\right) R = 0 \tag{3.11}$$

Identifying the term q/r as β_ϕ, it follows that $\beta_t^2 - q^2/r^2 = \beta_r^2$. Solving (3.10) results in

$$\Phi(\phi) = \begin{cases} \cos(q\phi + \alpha) \\ \sin(q\phi + \alpha) \end{cases} \tag{3.12}$$

where α is a constant phase shift. Again it will be noted that q must be an

integer, since it is required that the field be self-consistent on each rotation of ϕ through 2π. q is known as the angular or azimuthal mode number.

Equation (3.11) is a form of Bessel's equation. Its solution will be in terms of Bessel functions of the form

$$R(r) = \begin{cases} AJ_q(\beta_t r) + A'N_q(\beta_t r) & \beta_t \text{ real} \\ CK_q(|\beta_t| r) + C'I_q(|\beta_t| r) & \beta_t \text{ imaginary} \end{cases} \tag{3.13}$$

where J_q and N_q are ordinary Bessel functions of the first and second kind, and of order q, which apply to cases of real β_t. If β_t is imaginary, then the solution will consist of modified Bessel functions, K_q and I_q. Figures 3.3 and 3.4 depict the behavior of these functions. More detailed discussions of Bessel function properties are found in Appendix A.

The basic properties of a guided mode in the fiber should be similar to those found in the slab guide. In particular, it is expected that (1) the solution in the core will be oscillatory, exhibiting no singularities, and (2) the solution in the cladding will monotonically decrease as radius increases. The first condition suggests that the ordinary Bessel functions should apply in the core region, and so $\beta_{t1} = (n_1^2 k_0^2 - \beta^2)^{1/2}$ is required to be real. Only the J_q functions can be used, since the N_q functions are infinite at $r = 0$. The second condition indicates that the K_q functions would exhibit the proper variation in the cladding, with I_q ruled out. Therefore $\beta_{t2} = (n_2^2 k_0^2 - \beta^2)^{1/2}$ is required to be imaginary.

Normalized transverse propagation and attenuation constants are defined:

$$u = \beta_{t1} a = a(n_1^2 k_0^2 - \beta^2)^{1/2} \tag{3.14}$$

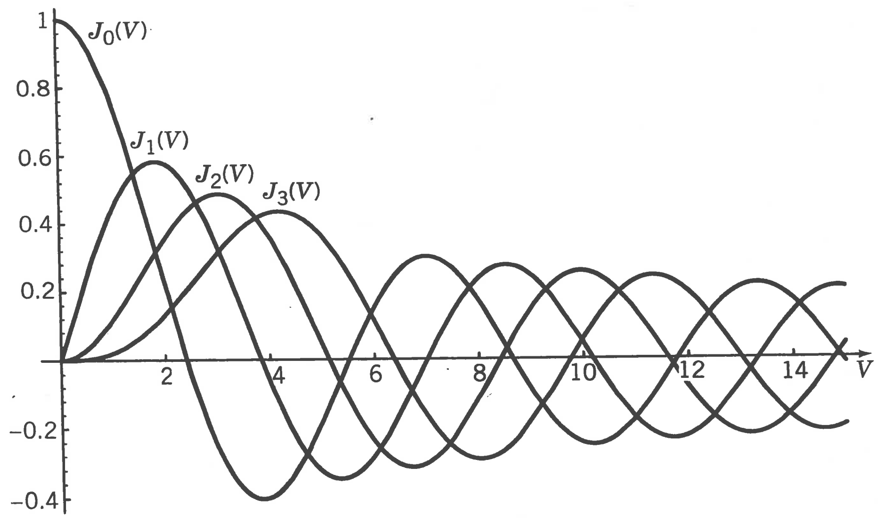
$$w = |\beta_{t2}| a = a(\beta^2 - n_2^2 k_0^2)^{1/2} \tag{3.15}$$

Setting $A' = C' = 0$ in (3.13), and using the $\sin(q\phi)$ dependence in (3.12) (with $\alpha = 0$), we find the complete solution for E_z:

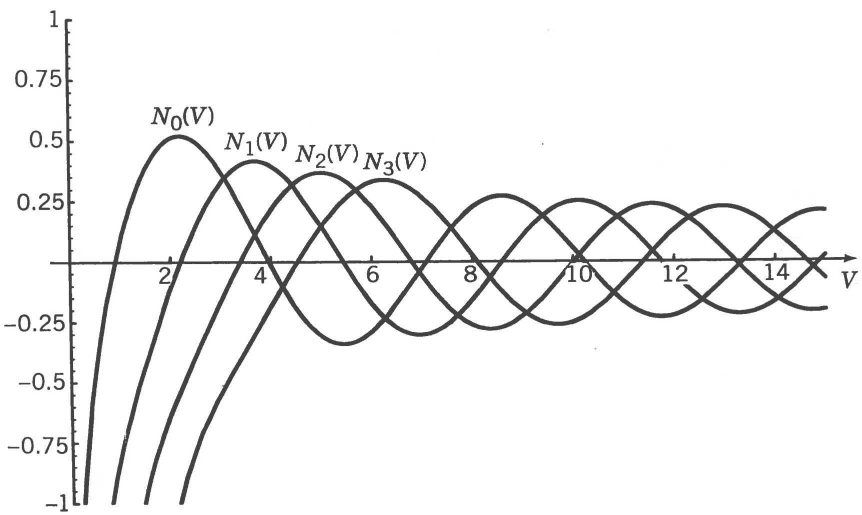
$$E_z = \begin{cases} AJ_q(ur/a)\sin(q\phi)\exp(-j\beta z) & r \leq a \\ CK_q(wr/a)\sin(q\phi)\exp(-j\beta z) & r \geq a \end{cases} \tag{3.16}$$

The integer requirement on q imposed by (3.12) carries over to the radial solution as well, thus limiting the Bessel functions to those of integer order. In similar fashion, the wave equation for $\mathbf{H}$ can be solved for the z component, leading to

$$H_z = \begin{cases} BJ_q(ur/a)\cos(q\phi)\exp(-j\beta z) & r \leq a \\ DK_q(wr/a)\cos(q\phi)\exp(-j\beta z) & r \geq a \end{cases} \tag{3.17}$$

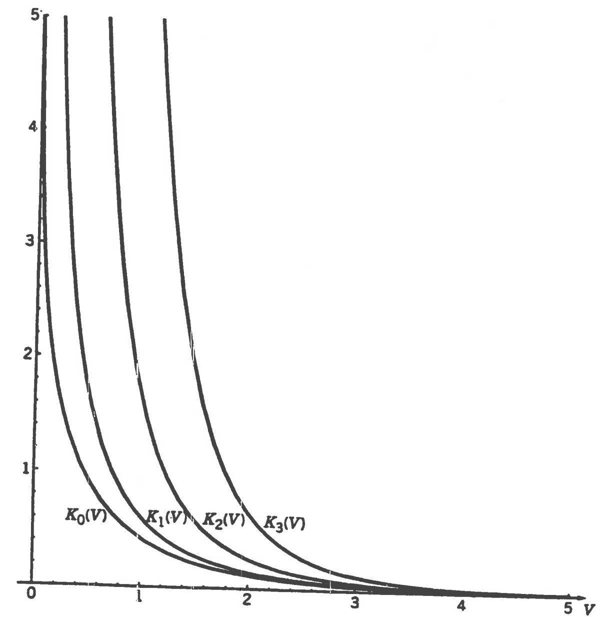


(a)

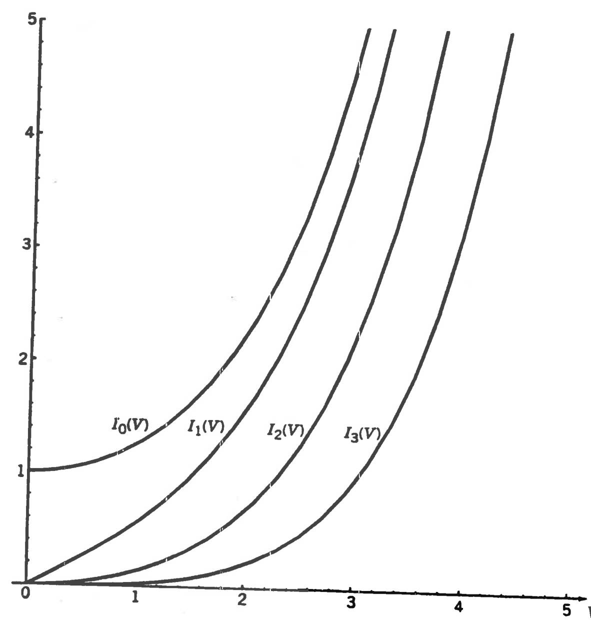


(b)

Figure 3.3. Ordinary Bessel functions.



(a)



(b)

Figure 3.4. Modified Bessel functions.

It will be seen that choosing the cosine dependence for H_z enables continuity of all tangential field components at $r = a$.

The transverse field components are found using the same method that was used with the slab guide; specifically, Maxwell's equations are used to find general expressions for all transverse components in terms of first derivatives of the z components. In cylindrical coordinates, it is found that

$$E_r = \frac{-j}{\beta_t^2}\left(\beta \frac{\partial E_z}{\partial r} + \omega\mu \frac{1}{r}\frac{\partial H_z}{\partial \phi}\right) \tag{3.18}$$

$$E_\phi = \frac{-j}{\beta_t^2}\left(\beta \frac{1}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu \frac{\partial H_z}{\partial r}\right) \tag{3.19}$$

$$H_r = \frac{-j}{\beta_t^2}\left(\beta \frac{\partial H_z}{\partial r} - \omega\epsilon \frac{1}{r}\frac{\partial E_z}{\partial \phi}\right) \tag{3.20}$$

$$H_\phi = \frac{-j}{\beta_t^2}\left(\beta \frac{1}{r}\frac{\partial H_z}{\partial \phi} + \omega\epsilon \frac{\partial E_z}{\partial r}\right) \tag{3.21}$$

Substituting (3.16) and (3.17) into (3.18) through (3.21) leads to the general expressions for all field components in the core and cladding regions. These are listed in Table 3.1 [1]. In the table, the primed functions, J_q' and K_q', indicate that their derivatives are taken with respect to the argument; for example,

$$J_q' = \frac{d}{d(ur/a)} J_q(ur/a).$$

3.3. MODE CLASSIFICATION AND THE EIGENVALUE EQUATIONS

The possible types of modes are understood by considering a few special cases. First, suppose that $q = 0$, and there is no z component of electric field ($A = C = 0$). In this case, Table 3.1 indicates that, in addition to H_z, the only nonzero field components will be E_ϕ and H_r. The *transverse electric* family of modes is thus obtained, designated TE_{0m}, where the subscript 0 indicates the value of the mode order, q; m is the mode rank, or radial mode number, which is greater than or equal to one. m has the same significance that it has in the slab guide, in which each value of m corresponds to one value each of k_{x1}, γ_2, and β. In the fiber a given m value is associated with a value each of u, w, and β. Transverse magnetic modes occur when $q = 0$ and when $H_z = 0$ ($B = D = 0$). Table 3.1 indicates that the only nonzero components in this case will be E_z, E_r, and H_ϕ. The subscripts in the designation TM_{0m} have the same meanings as in TE_{0m}. Modes of designation TE_{qm} and TM_{qm} where $q \neq 0$ are not possible. This is because too many field components would result, having too few constants

Table 3.1 General Step Index Fiber Fields in Phasor Form (exp ($-j\beta z$) Dependence Understood)

$r < a$:

$$E_z = AJ_q\left(\frac{ur}{a}\right)\sin(q\phi) \tag{3.22}$$

$$E_r = \left[-A\frac{j\beta}{(u/a)}J_q'\left(\frac{ur}{a}\right) + B\frac{j\omega\mu}{(u/a)^2}\frac{q}{r}J_q\left(\frac{ur}{a}\right)\right]\sin(q\phi) \tag{3.23}$$

$$E_\phi = \left[-A\frac{j\beta}{(u/a)^2}\frac{q}{r}J_q\left(\frac{ur}{a}\right) + B\frac{j\omega\mu}{(u/a)}J_q'\left(\frac{ur}{a}\right)\right]\cos(q\phi) \tag{3.24}$$

$$H_z = BJ_q\left(\frac{ur}{a}\right)\cos(q\phi) \tag{3.25}$$

$$H_r = \left[A\frac{j\omega\epsilon_0 n_1^2}{(u/a)^2}\frac{q}{r}J_q\left(\frac{ur}{a}\right) - B\frac{j\beta}{(u/a)}J_q'\left(\frac{ur}{a}\right)\right]\cos(q\phi) \tag{3.26}$$

$$H_\phi = \left[-A\frac{j\omega\epsilon_0 n_1^2}{(u/a)}J_q'\left(\frac{ur}{a}\right) + B\frac{j\beta}{(u/a)^2}\frac{q}{r}J_q\left(\frac{ur}{a}\right)\right]\sin(q\phi) \tag{3.27}$$

$r > a$:

$$E_z = CK_q\left(\frac{wr}{a}\right)\sin(q\phi) \tag{3.28}$$

$$E_r = \left[C\frac{j\beta}{(w/a)}K_q'\left(\frac{wr}{a}\right) - D\frac{j\omega\mu}{(w/a)^2}\frac{q}{r}K_q\left(\frac{wr}{a}\right)\right]\sin(q\phi) \tag{3.29}$$

$$E_\phi = \left[C\frac{j\beta}{(w/a)^2}\frac{q}{r}K_q\left(\frac{wr}{a}\right) - D\frac{j\omega\mu}{(w/a)}K_q'\left(\frac{wr}{a}\right)\right]\cos(q\phi) \tag{3.30}$$

$$H_z = DK_q\left(\frac{wr}{a}\right)\cos(q\phi) \tag{3.31}$$

$$H_r = \left[-C\frac{j\omega\epsilon_0 n_2^2}{(w/a)^2}\frac{q}{r}K_q\left(\frac{wr}{a}\right) + D\frac{j\beta}{(w/a)}K_q'\left(\frac{wr}{a}\right)\right]\cos(q\phi) \tag{3.32}$$

$$H_\phi = \left[C\frac{j\omega\epsilon_0 n_2^2}{(w/a)}K_q'\left(\frac{wr}{a}\right) - D\frac{j\beta}{(w/a)^2}\frac{q}{r}K_q\left(\frac{wr}{a}\right)\right]\sin(q\phi) \tag{3.33}$$

to enable satisfaction of all boundary conditions. Physically, the $q \neq 0$ case corresponds to a skew ray, which cannot maintain purely transverse electric and magnetic field components, as described in Section 3.1.

For cases in which $q \neq 0$, hybrid modes result, having z components of both **E** and **H**. These are designated either HE_{qm} or EH_{qm}, depending on the characteristics of the eigenvalue equation. Modes with designations HE_{0m} and EH_{0m} are not possible, again because these fields would be too simplified to enable all boundary conditions to be satisfied.

With the transverse field components, eigenvalue equations can be derived that determine β, u, and w for the modes. Following the method of Okoshi [1], use is made of the requirement that all field components that are tangent to the core–cladding boundary at $r = a$ be continuous across it. All ϕ and z components apply, producing four equations in all.

of the z components of electric and magnetic fields results in

$$AJ_q(u) - CK_q(w) = 0 \tag{3.34}$$

for $E_{z1}(a) = E_{z2}(a)$, and

$$BJ_q(u) - DK_q(w) = 0 \tag{3.35}$$

for $H_{z1}(a) = H_{z2}(a)$. Continuity of the ϕ components at $r = a$ results in

$$A \frac{j\beta}{(u/a)^2} \frac{qJ_q(u)}{a} - B \frac{j\omega\mu}{(u/a)} J'_q(u) + C \frac{j\beta}{(w/a)^2} \frac{qK_q(w)}{a} - D \frac{j\omega\mu}{(w/a)} K'_q(w) = 0 \tag{3.36}$$

for E_ϕ, and

$$A \frac{j\omega\epsilon_0 n_1^2}{(u/a)} J'_q(u) - B \frac{j\beta}{(u/a)^2} \frac{qJ_q(u)}{a} + C \frac{j\omega\epsilon_0 n_2^2}{(w/a)} K'_q(w) - D \frac{j\beta}{(w/a)^2} \frac{qK_q(w)}{a} = 0 \tag{3.37}$$

for H_ϕ. Equations (3.34) through (3.37) can be written in matrix form as

$$[M] \cdot \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \tag{3.38}$$

where $[M]$ is a matrix composed of the coefficients of A, B, C, and D in (3.34) through (3.37). For (3.38) to hold, the determinant of $[M]$ must vanish. Evaluating the determinant and requiring a zero result, we find [1]

$$\left(\frac{J'_q(u)}{uJ_q(u)} + \frac{K'_q(w)}{wK_q(w)} \right) \left(\frac{n_1^2}{n_2^2} \frac{J'_q(u)}{uJ_q(u)} + \frac{K'_q(w)}{wK_q(w)} \right) = q^2 \left(\frac{1}{u^2} + \frac{1}{w^2} \right) \left(\frac{n_1^2}{n_2^2} \frac{1}{u^2} + \frac{1}{w^2} \right) \tag{3.39}$$

Equation (3.39) is greatly simplified by the *weakly guiding approximation*, which states that $n_1 \approx n_2$. This condition is almost always satisfied in practical optical fibers. With the approximation, (3.39) becomes

$$\left(\frac{J'_q(u)}{uJ_q(u)} + \frac{K'_q(w)}{wK_q(w)} \right) = \pm q \left(\frac{1}{u^2} + \frac{1}{w^2} \right) \tag{3.40}$$

The case in which $q = 0$ applies to TE and TM modes. Using Bessel function identities, (3.40) becomes

$$u \frac{J_0(u)}{J_1(u)} = -w \frac{K_0(w)}{K_1(w)} \qquad TE_{0m}, TM_{0m} \tag{3.41}$$

For $q > 0$, the right-hand side of (3.40) will be either positive or negative. Equation (3.40) with positive right-hand side is the eigenvalue equation for EH_{qm} modes; with negative right-hand side, it is the equation for HE_{qm} modes.

The EH and HE designations originate from an early system in which comparisons are made of E_z and H_z to the transverse field components at some reference point [2]. This system is somewhat arbitrary, since results depend on the choice of the reference point. The EH or HE nomenclature is nevertheless convenient and descriptive in classifying modes according to the above cases that occur in (3.40).

For $q > 0$, (3.40) can be specialized for the two cases of positive and negative right-hand side by again using Bessel function identities. The results are

$$u \frac{J_q(u)}{J_{q+1}(u)} = -w \frac{K_q(w)}{K_{q+1}(w)} \qquad EH_{qm}\ (q \geq 1) \tag{3.42}$$

$$u \frac{J_q(u)}{J_{q-1}(u)} = w \frac{K_q(w)}{K_{q-1}(w)} \qquad HE_{qm}\ (q \geq 1) \tag{3.43}$$

As a further simplification, (3.41) to (3.43) can be written as a single equation by first rewriting (3.43) as

$$u \frac{J_{q-2}(u)}{J_{q-1}(u)} = -w \frac{K_{q-2}(w)}{K_{q-1}(w)} \qquad HE_{qm}\ (q \geq 1) \tag{3.44}$$

Defining a new integer variable, l, (3.41), (3.42), and (3.44) can be expressed as [1]

$$u \frac{J_{l-1}(u)}{J_l(u)} = -w \frac{K_{l-1}(w)}{K_l(w)} \qquad l = \begin{cases} 1 & TE_{0m},\ TM_{0m} \\ q+1 & EH_{qm} \\ q-1 & HE_{qm} \end{cases} \tag{3.45}$$

The significance of (3.45) is that a set of degenerate EH, HE, TE, or TM modes will combine to form a new "composite" mode whose eigenvalue equation is (3.45). It will be shown in a later section that the electric field superposition of the combining modes is linearly polarized in the transverse plane; for this reason, the composite modes are referred to as the LP_{lm} modes. The LP system is a simplified way of describing modes in weakly guiding fibers. Furthermore, the LP modes are almost always observed in practice and are readily identified by their transverse intensity profiles.

3.4. SOLUTIONS OF THE EIGENVALUE EQUATIONS

A graphical solution method for the eigenvalue equations is demonstrated here for the cases of TE_{0m}, TM_{0m} (LP_{1m}), and HE_{1m} (LP_{0m}) modes [3]. To begin, the normalized frequency parameter, V, is defined as

$$V = (u^2 + w^2)^{1/2} = ak_0(n_1^2 - n_2^2)^{1/2} = n_1 a k_0 \sqrt{2\Delta} \tag{3.46}$$

This important parameter, often called the V number, embodies the fiber structural parameters and frequency. It is used with (3.45) to determine cutoff conditions for the modes, propagation constants, and information on power confinement. It is seen that V is directly analogous to the radius function R, defined in (2.46), that is used in the graphical solution for the slab waveguide.

Consider first (3.41), which can be rewritten to read

$$\frac{J_1(u)}{J_0(u)} = -\frac{u}{w}\frac{K_1(w)}{K_0(w)} \tag{3.47}$$

Noting the behavior of J_0 and J_1 as shown in Fig. 3.3, it can be seen that the left-hand side of (3.47), when plotted as a function of u, will produce curves that resemble those of a tangent function. These are shown in the two graphs that comprise Fig. 3.5. In the graphs, the asymptotes and zeros correspond to appropriate zeros in Fig. 3.3. The right-hand side of (3.47) is plotted as the curve shown in the lower half-plane of each graph. Its intersections with the J_1/J_0 curves indicate solutions to (3.47). The values of u thus obtained can be used to find w and β, knowing V. The first plot was made using $V = 2$. From (3.46), the maximum value of u for a given V occurs when $w = 0$, at which $u = V$. A vertical line is drawn on the first graph at $u = 2$ that marks this position; it thus indicates the upper bound on the solutions for u that will appear on the graph. The second graph was plotted for $V = 8$, showing the vertical boundary at $u = 8$.

HE_{1m} (LP_{0m}) modes can be exhibited on the same graph by first rewriting

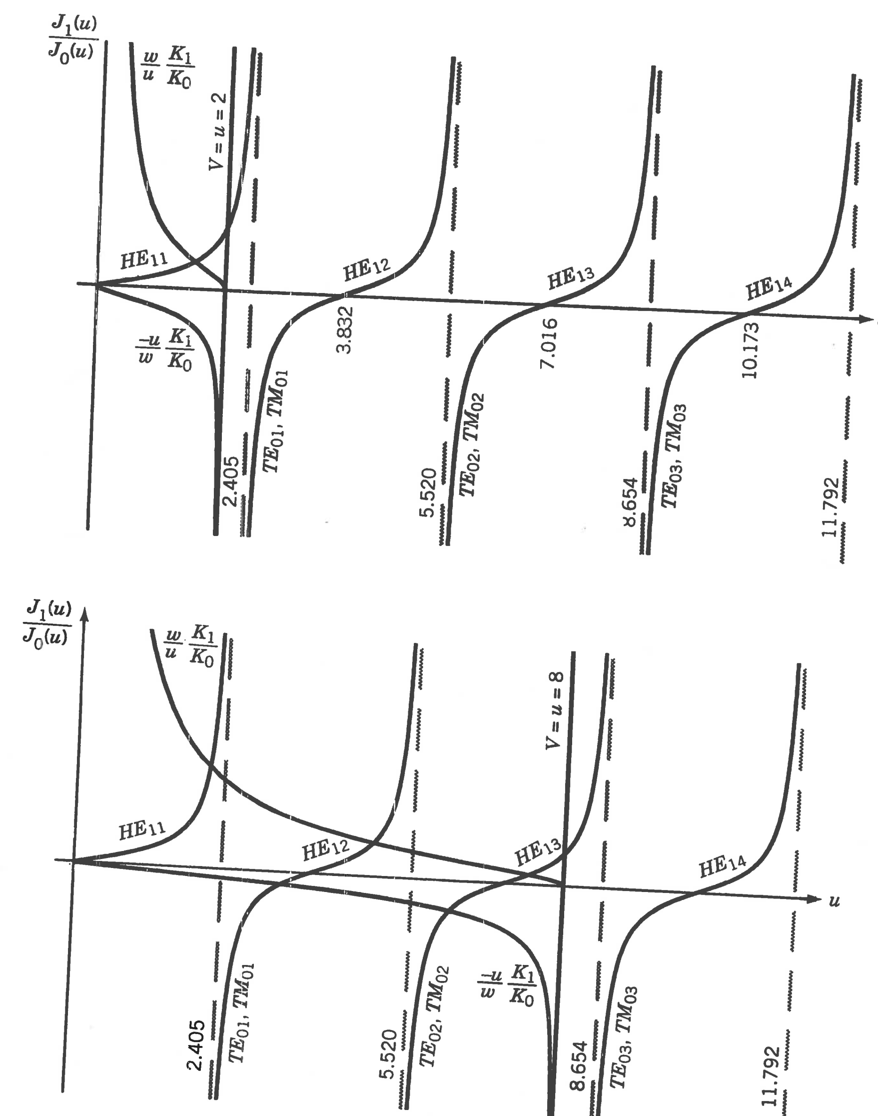


Figure 3.5: Graphical solutions of Eqs. (3.47) and (3.48) for $V = 2$ (upper) and $V = 8$ (lower).

(3.43) to read

$$\frac{J_1(u)}{J_0(u)} = \frac{w}{u}\frac{K_1(w)}{K_0(w)} \tag{3.48}$$

The left-hand side of (3.48) produces the same curves as the left-hand side of (3.47). The right-hand side of (3.48) produces th shown in the upper half-plane of each graph in Fig. 3.5. The fact that this curve approaches infinity as u becomes small is evident from the equation and by inspection of Fig. 3.4. The fact that the function reaches zero at $V = u$ is not obvious but can be understood by noting the behavior of the modified Bessel functions for small w, where

$$\frac{w}{u}\frac{K_1(w)}{K_0(w)} \approx \frac{1}{u\ln(2/1.8w)} \qquad (3.49)$$

The above approaches zero as w approaches zero.

The branches of $J_1(u)/J_0(u)$ in Fig. 3.5 are labeled by their mode designations, where the values of m on a given branch will differ for the two mode types. A change in frequency or numerical aperture will change V and will thus change the position of the vertical "V line" on the graph accordingly. The curves that correspond to the right-hand sides of (3.47) and (3.48) either stretch or contract with the vertical line position, thus producing fewer or more solutions, depending on whether V decreases or increases.

A few important points should be noted.

1. If the vertical line is positioned at a zero of a $J_1(u)/J_0(u)$ curve, then the cutoff point is reached for the HE_{1m} (LP_{0m}) mode associated with that branch.
2. If the vertical line is positioned at a lower asymptote of a $J_1(u)/J_0(u)$ curve, then the cutoff point is reached for the TE_{0m} or TM_{0m} (LP_{1m}) mode corresponding to that branch number.
3. As V increases, all modes having $u < V$ will continue to have solutions and will thus continue to propagate.
4. There is no cutoff for the HE_{11} (LP_{01}) mode. The fiber thus propagates this mode only for values of V in the range $0 < V < 2.405$. The upper graph of Fig. 3.5 shows a case of single-mode operation.
5. Values of w for a given mode can achieve any value between zero and infinity.
6. The values of u are bounded for a given mode. This fact can be deduced from the graph or from (3.45) by noting that, at cutoff ($w = 0$), u will be the mth zero of J_{l-1} (this does not include the zero at $u = 0$, except for the case when $l = 0$). Far above cutoff, as $w \to \infty$, u will approach the mth zero of J_l. For example, u values for the TE_{01} mode occur within the range $2.405 \le u \le 3.832$.
7. Values of V for a given mode will occur over the range $V_c \le V \le \infty$, where V_c is the value of V at cutoff for that mode.

Similar plots can be made for HE_{qm} modes for $q \ge 2$, and for EH_{qm} modes. These will produce curves of J_l/J_{l-1} that will fall between some of the curves shown in the graphs, making the progression of mode appearance with increasing V somewhat more complicated than is depicted there. Nevertheless, single mode operation will still occur over the range $0 < V < 2.405$.

3.5. CUTOFF CONDITIONS

Cutoff for a given mode can be determined directly from (3.45) by setting $w = 0$ which means that $u = V$. Under this condition, the cutoff condition is obtained from (3.45):

$$\frac{VJ_{l-1}(V)}{J_l(V)} = 0 \qquad (3.50)$$

The above will be satisfied for all values of V that correspond to zeros of J_{l-1}. The special case of $V = 0$ is handled by the small argument approximation of (3.50):

$$\frac{VJ_{l-1}(V)}{J_l(V)} \approx \frac{lV(V/2)^{l-1}}{(V/2)^l} = 2l \qquad (3.51)$$

The above will be zero only when $l = 0$. Hence, as V approaches zero, (3.50) is satisfied only by the special case

$$\frac{VJ_{-1}(V)}{J_0(V)} = 0 \qquad (3.52)$$

where $J_{-1}(V) = -J_1(V)$. Equation (3.52) thus applies to the HE_{11} (LP_{01}) mode, having cutoff at $V = 0$.

For the remaining zeros, a simplified form of (3.50) is used:

$$J_{l-1}(V) = 0 \qquad (3.53)$$

It will be recalled that $l = 1$ for TE and TM modes, $q + 1$ for EH modes, and $q - 1$ for HE modes. Using (3.53), each Bessel function zero is considered, and the modes with cutoff occurring at that value of V are established. For example, 2.405 is the first zero of J_0. Equation (3.53) becomes $J_0(2.405) = 0$, which indicates that $l = 1$. Considering the above relations between l and q, the modes TE_{01}, TM_{01}, and HE_{21} will all achieve cutoff at $V = 2.405$. The value of m is one for these modes because $V = 2.405$ is the first instance in which any of these modes (with the given value of q) occur as V increases. From (3.45), it is seen that these modes will all have the same eigenvalue equation, which means that they will be *degenerate*; that is, they will all have the same value