Optics 554 — Homework 2
Due Wednesday, February 19, 2021

This problem involves the principle of reciprocity in Optics, and Fraunhofer diffraction. The purpose of a telescope array is to achieve higher resolution. We will make here the approximation that we have a single-dimensional problem. The telescope is operating at a frequency of 3 GHz. Let us consider an array made of antennas of 2 m long. The antennas are disposed symmetrically. One is at the center. The next two are 10 meter away from the central one (distance center to center). The last two are at 30 meter from the central one (center to center). We thus have a total of 5 antennas.

1. Considering only the central antenna as a square aperture, find the corresponding angular resolution (of the telescope having only one antenna). (consider this to be a Fraunhofer diffraction pattern problem)
2. Give you best numerical estimate of the resolution of the array consisting of 5 antennas.
3. What are the analogies and differences between the telescope array and a diffraction grating?

Solutions

Fraunhofer approximation, in one dimension, is:

$$\begin{aligned}
&\rho(r - r_0) \approx (z - \frac{k_x}{x}x) \\
&K(\theta) = 1 \\
&\rho(r - r_0) = 1 \text{in the denominator (amplitude term)}
\end{aligned} \tag{1}$$

The one-dimensional diffraction pattern is thus given by:

$$E(x, L) \approx \sqrt{\frac{i}{L\lambda}} e^{-ikL} \int_{-\infty}^{\infty} E(x) e^{ik_x x} dx. \tag{2}$$

In terms of the field amplitude; substituting for $E(x, z)$ $\mathcal{E}(x, z)e^{-ikz}$:

$$\tilde{\mathcal{E}}(x, L) = \sqrt{\frac{i}{L\lambda}} \int_{-\infty}^{\infty} \mathcal{E}(x) e^{ik_x x} dx = \sqrt{\frac{i}{L\lambda}} \tilde{\mathcal{E}} k_x. \tag{3}$$

For a "square aperture" of width D, the Fourier transform is the sinc function:

$$\tilde{\mathcal{E}}(k_x) = \mathcal{E}_0 D \frac{\sin \frac{k_x D}{2}}{\frac{k_x D}{2}}. \tag{4}$$

In this particular case, we are going to the limit $L \to \infty$. The relevant variable in this problem is not k_x but $\theta = k_x/k$. The far field angular distribution is thus of the form:

$$\tilde{\mathcal{E}}(\theta) \propto \frac{\sin \frac{\pi D \theta}{\lambda}}{\frac{\pi D \theta}{\lambda}}. \tag{5}$$

One of the numerous possible definitions of the resolution could be half the angular spacing between the zeros, which is θ_w given by:

$$\frac{\pi D \theta_w}{\lambda} = \pi, \tag{6}$$

or:

$$\theta_w = \frac{\lambda}{D} = \frac{0.1}{2} = 0.05 radian \tag{7}$$

which is a lousy resolution. This is for the field — one could divide by a factor 2 for the intensity.

For the combination of the 5 apertures, the Fourier transform is:

$$\tilde{\mathcal{E}}(k_x) = \mathcal{E}_0 D \frac{\sin \frac{k_x D}{2}}{\frac{k_x D}{2}} \left[1 + 2\cos(10k_x) + 2\cos(30k_x)\right]. \tag{8}$$