Optics

Optics 554 — Homework 1

Fresnel - Fraunhofer diffraction problem.
Due Wednesday, February 10, 2021

Consider Gaussian beam propagation as a diffraction problem.

1. Consider a plane wave with a Gaussian intensity distribution of waist w_0 (as could be obtained from an infinite uniform plane wave, normally incident on an apodized aperture, or simply a Gaussian beam emerging from the flat output coupler of a laser). Write the general Huygens integral for the field at a distance z from this aperture.

2. Consider only one transverse coordinate to find the radial distribution of the electrical field amplitude $\mathcal{E}(x, z)$, using the Fraunhofer approximation.

3. Consider only one transverse coordinate to find the radial distribution of the electrical field amplitude $\mathcal{E}(x, z)$, using the Fresnel approximation.

4. Compare the results of (2) and (3) with that of Gaussian beam propagation, and explain the differences/similarities.

Solutions

Question 1

Each point of the "aperture" is considered to be a source of spherical wave issued from P_0.

$$E = \frac{E_0 e^{ik\rho(r-r_0)}}{\rho(r-r_0)}. \tag{1}$$

Huygens's integral is the sum of these spherical waves, with a normalization factor $i/(L\lambda)$, and an "obliquity factor $K[\theta(r, r_0)]$":

$$E(x,y,z) = \frac{i}{L\lambda}\int\int_{S_0} \frac{E(x_0,y_0,z_0)e^{-ik\rho(r-r_0)}}{\rho(r-r_0)}K(\theta)dS_0. \tag{2}$$

Question 2

Fraunhofer approximation, in one dimension, is:

$$\begin{aligned}
&\rho(r-r_0) \approx (z - \frac{k_x}{x}x)\\
&K(\theta) = 1\\
&\rho(r-r_0) = 1 \text{in the denominator (amplitude term)}
\end{aligned} \tag{3}$$

The one-dimensional diffraction pattern is thus given by:

$$E(x,L) \approx \sqrt{\frac{i}{L\lambda}}e^{-ikL}\int_{-\infty}^{\infty} E(x)e^{ik_x x}dx. \tag{4}$$

In terms of the field amplitude; substituting for $E(x,z)$ $\mathcal{E}(x,z)e^{-ikz}$:

$$\mathcal{E}(x,L) = \sqrt{\frac{i}{L\lambda}}\int_{-\infty}^{\infty} \mathcal{E}(x)e^{ik_x x}dx = \sqrt{\frac{i}{L\lambda}}\tilde{\mathcal{E}}k_x. \tag{5}$$

For the initial Gaussian field issued from the beam waist: $\mathcal{E}_0 = \exp(-x^2/w_0^2)$, the Fourier transform is:

$$\tilde{\mathcal{E}}(k_x) = e^{-\frac{k_x^2 w_0^2}{4}} = e^{-\frac{k^2\theta^2 w_0^2}{4}} = e^{-\frac{k^2 w_0^2 x^2}{4z^2}} = e^{-\frac{x^2}{w^2}}. \tag{6}$$

The transformed beam waist is thus:

$$w^2 = \frac{k^2 w_0^2}{4z^2} = w_0^2 \left(\frac{\lambda}{\pi w_0^2}\right)^2 z^2, \tag{7}$$

which is indeed the expression for a Gaussian beam, in the approximation $z \gg \pi w_0^2/\lambda$.

Question 3

Fresnel approximation, in one dimension, consists in making in the Huygen's integral the substitution:

$$\begin{aligned} &K(\theta) \approx 1 \\ &\rho(r - r_0) \approx (z - z_0) + \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)}. \end{aligned} \tag{8}$$

The Fresnel integral to solve in one dimension is thus:

$$\begin{aligned} \tilde{\mathcal{E}}(x_2) &= \sqrt{\frac{i}{L\lambda}} \int_{-\infty}^{\infty} \mathcal{E}(x_1) e^{-ik(x_2 - x_1)^2/2L} dx_1 \\ &= \sqrt{\frac{i}{L\lambda}} \int_{-\infty}^{\infty} e^{-\frac{x_1^2}{w_0^2}} e^{-\frac{ik(x_2 - x_1)^2}{2L}} dx_1. \end{aligned} \tag{9}$$

The result of the integration is also a Gaussian, with a term proportional to x_2^2 in the exponent. Identifying this Gaussian with a real part of $\exp(-x_2^2/w^2)$, we find that w is given by:

$$\frac{1}{w^2} = \mathrm{Re}\left[\frac{ik}{2L} + \frac{k^2}{4L^2}\left(\frac{1}{\frac{1}{w_0^2} + \frac{ik}{2L}}\right)\right]. \tag{10}$$

$$\frac{1}{w^2} = \frac{k^2}{4L^2}\left(\frac{\frac{1}{w_0^2} - \frac{ik}{2L}}{\frac{1}{w_0^2} + \frac{k^2}{4L^2}}\right)$$

This leads to the well known result:

$$w^2 = w_0^2\left[1 + \left(\frac{\lambda}{\pi w_0^2}\right)^2 L^2\right] \tag{11}$$