Optics 463 — Homework 9 -Fraunhofer
due Tuesday, November 13, 2018

1 Diffraction of a Gaussian beam

1.1 Fraunhofer approximation

Consider a plane wave with a Gaussian intensity distribution of waist w_0 (as could be obtained from an infinite uniform plane wave, normally incident on an apodized aperture, or simply a Gaussian beam emerging from the flat output coupler of a laser). Calculate the diffracted field $\mathcal{E}(x, z)$ using the Fraunhofer approximation, considering only one transverse coordinate to find the radial distribution of the electrical field amplitude. Compare the results with that of Gaussian beam propagation, and explain the differences/similarities.

1.2 Paraxial approximation

Same as above, but using the paraxial approximation.

2 Telescope array

This problem involves the principle of reciprocity in Optics, and Fraunhofer diffraction. The purpose of a telescope array is to achieve higher resolution. We will make here the approximation that we have a single-dimensional problem. The telescope could be optical or RF. Let us consider an array made of antennas of 2 m long. The antennas are disposed symmetrically. One is at the center. The next two are 10 meter away (distance center to center). The last two are at 30 meter from the central one (center to center). We thus have a total of 5 antennas. The frequency of the radiation is 3 GHz.

1. Considering only the central antenna as a square aperture, find the corresponding angular resolution (of the telescope having only one antenna). (consider this to be a Fraunhofer diffraction pattern problem)

2. Give your best numerical estimate of the resolution of the array consisting of 5 antennas.

3. How will the result of the previous two parts be changed if the radiation is $3 \times 10^{14} Hz$?

Solutions

Question 1

Fraunhofer diffraction is the Fourier transform of the initial field: In one dimension:

$$\mathcal{E}(x, L) \propto \int_{-\infty}^{\infty} \mathcal{E}(x) e^{ik_x x} dx = \tilde{\mathcal{E}}(k_x). \tag{1}$$

For the initial Gaussian field issued from the beam waist: $\mathcal{E}_0 = \exp(-x^2/w_0^2)$, the Fourier transform is:

$$\tilde{\mathcal{E}}(k_x) = e^{-\frac{k_x^2 w_0^2}{4}} = e^{-\frac{k^2\theta^2 w_0^2}{4}} = e^{-\frac{k^2 w_0^2 x^2}{4z^2}} = e^{-\frac{x^2}{w^2}}. \tag{2}$$

The transformed beam waist is thus:

$$w^2 = \frac{k^2 w_0^2}{4z^2} = w_0^2 \left(\frac{\lambda}{\pi w_0^2}\right)^2 z^2, \tag{3}$$

which is indeed the expression for a Gaussian beam, in the approximation $z \gg \pi w_0^2/\lambda$.

Question 2

We have already the Fourier transform of the field distribution:

$$\tilde{\mathcal{E}}(0, k_x) \propto e^{-\frac{k_x^2 w_0^2}{4}}. \tag{4}$$

Propagating the field:

$$\tilde{\mathcal{E}}(z, k_x) = \tilde{\mathcal{E}}(0, k_x) e^{ik_x^2 z/2k} \tag{5}$$

The exponential factor is thus:

$$-\frac{k_x^2 w_0^2}{4}\left(1 - i\frac{z}{2k}\frac{4}{w_0^2}\right) - \frac{k_x^2 w_0^2}{4}\left(1 - i\frac{z}{\rho_0}\right), \tag{6}$$

where ρ_0 is our usual Rayleigh range $\pi w_0 * 2/\lambda$. The inverse Fourier transform of

$$e^{-(\Omega^2\tau^2/4)(1+ia)}$$

being:

$$e^{-\frac{t^2}{\tau^2(1+a^2)}}$$

, we find that, after substitution, the inverse Fourier transform of Eq. (5) is:

$$e^{-\frac{x^2}{w_0^2(1+z^2/\rho_0^2)}}. \tag{7}$$

There is also a phase factor that would correspond to the radius of curvature of the wavefront.

3 Telescope array

Solutions

Fraunhofer approximation, in one dimension, is:

$$\begin{aligned}
&\rho(r - r_0) \approx (z - \frac{k_x}{x}x) \\
&K(\theta) = 1 \\
&\rho(r - r_0) = 1 \text{in the denominator (amplitude term)}
\end{aligned} \tag{8}$$

The one-dimensional diffraction pattern is thus given by:

$$E(x, L) \approx \sqrt{\frac{i}{L\lambda}} e^{-ikL} \int_{-\infty}^{\infty} E(x) e^{ik_x x} dx. \tag{9}$$

In terms of the field amplitude; substituting $\mathcal{E}(x, z)e^{-ikz}$ for $E(x, z)$:

$$\tilde{\mathcal{E}}(x, L) = \sqrt{\frac{i}{L\lambda}} \int_{-D/2}^{D/2} \mathcal{E}(x) e^{ik_x x} dx = \sqrt{\frac{i}{L\lambda}} \tilde{\mathcal{E}}(k_x). \tag{10}$$

For a “square aperture” of width D, the Fourier transform is the sinc function:

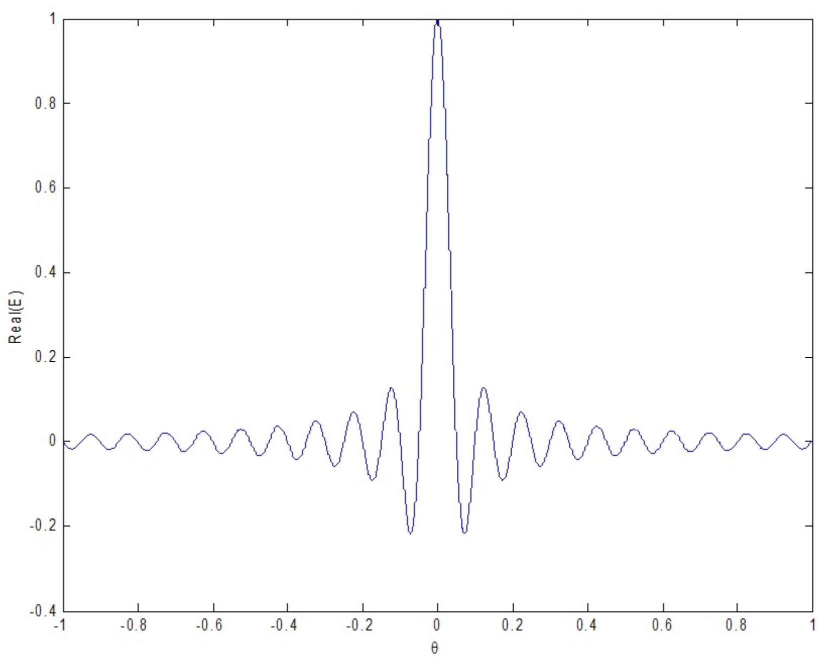

Figure 1: Plot of the real electric field for $\lambda = 0.1$ m and one telescope.

$$\tilde{\mathcal{E}}(k_x) = \mathcal{E}_0 D \frac{\sin \frac{k_x D}{2}}{\frac{k_x D}{2}}. \tag{11}$$

In this particular case, we are going to the limit $L \to \infty$. The relevant variable in this problem is not k_x but $\theta = k_x/k$. The far field angular distribution is thus of the form:

$$\tilde{\mathcal{E}}(\theta) \propto \frac{\sin \frac{\pi D \theta}{\lambda}}{\frac{\pi D \theta}{\lambda}}. \tag{12}$$

One of the numerous possible definitions of the resolution could be half the angular spacing between the zeros, which is θ_w given by:

$$\frac{\pi D \theta_w}{\lambda} = \pi, \tag{13}$$

or:

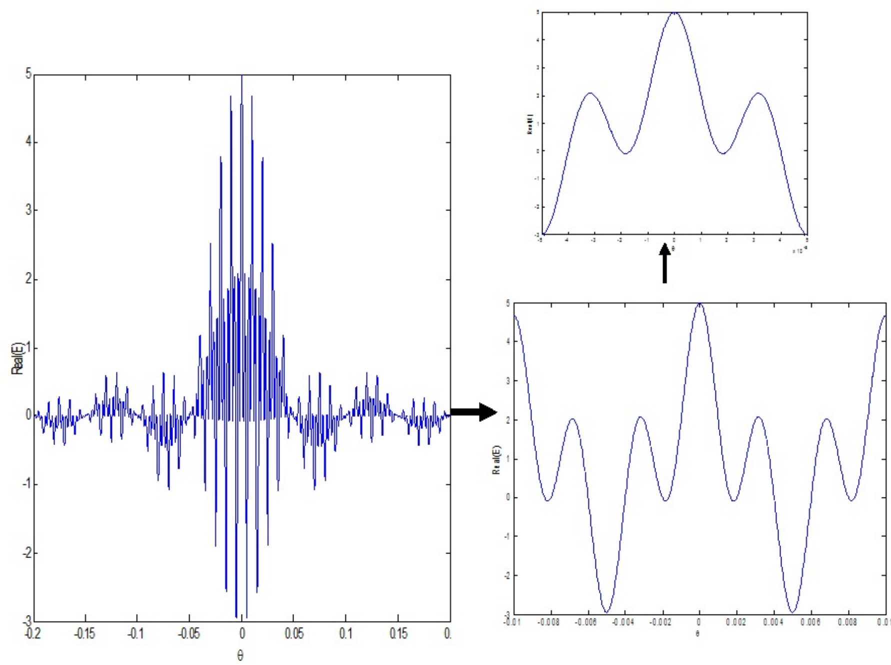

Figure 2: Plot of the real electric field for $\lambda = 0.1$ m and five telescopes.

$$\theta_w = \frac{\lambda}{D} = \frac{0.1}{2} = 0.05\text{radian} \tag{14}$$

which is a lousy resolution. This is for the field — one could divide by a factor 2 for the intensity.

For the combination of the 5 apertures, the Fourier transform is:

$$\tilde{\mathcal{E}}(k_x) = \mathcal{E}_0 D \frac{\sin \frac{k_x D}{2}}{\frac{k_x D}{2}} \left[1 + 2\cos(10k_x) + 2\cos(30k_x)\right]. \tag{15}$$

Which can be written as

$$\tilde{\mathcal{E}}(\theta) \propto \frac{\sin \frac{\pi D \theta}{\lambda}}{\frac{\pi D \theta}{\lambda}} \left[1 + 2\cos(20\pi\theta/\lambda) + 2\cos(60\pi\theta/\lambda)\right]. \tag{16}$$

From figure 2, the angular resolution can be found $\theta = 0.0018$ rad. If we use the formula (6) for a D which covers the total spread of telescopes $D = 62$ m the resolution angle is $\theta = 0.0016$ rad which is a very good approximation. We can simply estimate the resolution of a set of telescopes by one telescope spanning the whole size covered by the set. By going to higher frequencies $\lambda = 100nm$,the resolution for one telescope is $\theta = 0.05 \times 10^{-5} rad$ and for five telescopes is $\theta = 0.0018 * 10^{-5} rad$.