Optics 554 — Homework 3

Gratings and Fraunhofer diffraction problem.

Due Wednesday, February 17, 2021

1) Resolution of a Grating

Consider a square grating of 600 lines/mm, 5 cm $\times$ 5 cm.

Questions:

(a) What is the blaze angle in Littrow configuration *in third order* at a wavelength of 500 nm?

(b) Assuming the grating is curved with a focal distance of 2 meter. What is the maximum resolution $\Delta\lambda$ that can be achieved in third order? You will assume that the beam is Gaussian, with a "diameter" $2w$ covering the dimension of the grating, and that the angle of incidence on the grating is 30 degrees. Explain/justify your approximation(s), if any.

(c) Keeping the angle of incidence of (b): At which wavelength will the first order diffraction angle coincide with the third order? What is the resolution $\Delta\lambda$ at that wavelength?

Solution

The grating formula gives:

$$\sin\alpha + \sin\beta = 2\sin\beta = 3\frac{\lambda}{d} = \frac{3. \times 0.5 \cdot 10^{-3} \times 600}{1} \tag{1}$$

which gives $\sin\beta = 0.45$ and the blaze angle is 26.7 degrees.

Question b

The third order diffraction angle is:

$$\sin\beta = 3\frac{\lambda}{d} - \sin 30^{\circ} = 0.9 - 0.5 = 0.4$$

hence $\beta = 23.6$ degrees.

Considering the grating aperture (Gaussian beam diffraction) Considering the angle of incidence β, the cross section seen by the beam is $\ell\cos\beta = 5\text{cm}\times 0.92 = 4,58$ cm. We thus have $w = 4.58/2 = 2.29$ cm. The focusing angle θ is thus:

$$\theta = \frac{\lambda}{\pi w_0} = \frac{w}{f} = (2.29/2)\cdot 10^{-2} \tag{2}$$

from which we extract $w_0 = 27.8\mu$m. The smallest angle that can be resolved is approximately $\Delta\beta = w_0/f = 14$ microradian. Taking the derivative of the grating equation:

$$\cos\beta\Delta\beta = 3\frac{\Delta\lambda}{d}$$

yielding the resolution:

$$\Delta\lambda = \frac{d\cos\beta}{3}\Delta\beta = 7\cdot 10^{-6}\mu\text{m}$$

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Alternate approach: taking the far field (Fraunhofer) of the grating Let us assume that the central element of the grating has a Fourier transform: $f(k_x d)$, If we have N elements, the Fourier transform for the whole grating is the sum:

$$\begin{aligned} f(k_x d) S = f(k_x d) & \Big[1 + e^{-ik_x a} + e^{-2k_x a} + \ldots e^{-(\frac{N-1}{2})k_x a} \\ + & \; e^{ik_x a} + e^{2k_x a} + \ldots e^{(\frac{N-1}{2})k_x a}\Big] \end{aligned} \tag{3}$$

The sum can be evaluated by taking the difference of:

$$\begin{aligned} S e^{ik_x a} &= \Big[1 + e^{-ik_x a} + e^{-2k_x a} + \ldots e^{-(\frac{N-3}{2})k_x a} \\ &\quad + e^{ik_x a} + e^{2k_x a} + \ldots e^{(\frac{N+1}{2})k_x a}\Big] \\ S e^{-ik_x a} &= \Big[1 + e^{-ik_x a} + e^{-2k_x a} + \ldots e^{-(\frac{N+1}{2})k_x a} \\ &\quad + e^{ik_x a} + e^{2k_x a} + \ldots e^{(\frac{N+3}{2})k_x a}\Big] \end{aligned} \tag{4}$$

which gives:

$$S = \frac{\sin \frac{N+3}{2} k_x a}{\sin k_x a} \tag{5}$$

The resolution is given roughly by the condition $N k_x a/2 = \pi$. Since $Na = W$ is the width of the grating, and $k_x = (2\pi/\lambda)\theta_x$, we find that θ_x is simply of the order of λ/W, which is the diffraction formula.

Question c For $\beta = 23.6^{\text{o}}$ [see (b)], $\sin\beta = 0.4$. The first order diffraction equation:

$$\sin\beta + \sin 30^{\text{o}} = \frac{\lambda}{d} = 0.4 + 0.5 = 0.9 = 0.6 \times \lambda$$

therefore, $\lambda = (1/0.6) \times 0.9 = 1.5\mu\text{m}$

Resolution: Taking the derivative of the grating equation:

$$\cos\beta \Delta\beta = \frac{\Delta\lambda}{d}$$

yielding the resolution:

$$\Delta\lambda = d \cos\beta \Delta\beta = (1/0.6) \times 0.916 \times 14 \cdot 10^{-6} = 21 \cdot 10^{-6}$$

. As expected, 3 $\times$ worse than in third order.

Fraunhofer Diffraction

2) Diffraction orders

Let us consider a transmission grating made of successive identical triangular transmission functions. The transmission function of one element of this one dimensional aperture is shown in Fig. 1.

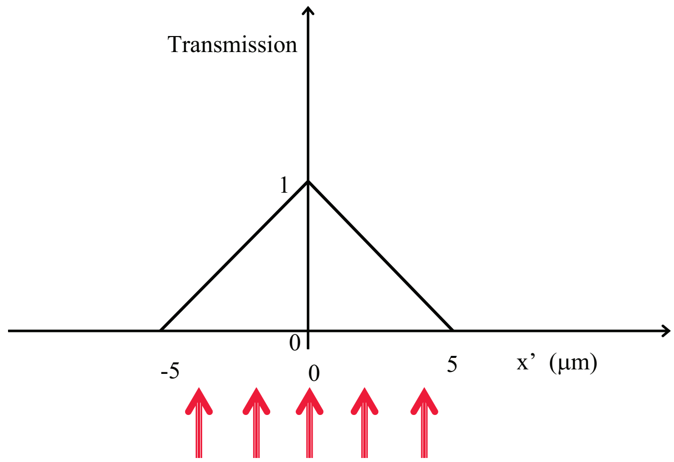

Figure 1: Triangular transmission function, with a peak transmission of 1 at $x' = 0$, and total opacity at $x' = \pm 5\mu$m. The axis x' is labeled in μm.

(a) Find the diffraction pattern — in intensity — for a uniform plane wave incident (normal incidence) on one element of this slide (as represented in Fig. 1), as it will appear on a screen situated at a distance of 1 meter from the slide. The wavelength is 1 μm. What is the full with at half maximum (FWHM) of the intensity pattern in the diffraction plane. (Hint: use the convolution theorem: the triangle is the autoconvolution of ??))

(b) What diffraction orders can be mainly observed with this grating? Find the diffraction efficiency in the orders +1 and -1 (Assuming the grating to be perfect, calculate the diffraction intensity in first order, and divide by the incident intensity)

Solution

The triangular element is the auto-convolution of a rectangle of height 0.2 and width 5. Aside from the amplitude factor, the Fourier transform of the triangle is the square of the Fourier transform of the rectangle, hence the square of a sinc:

$$\mathcal{E}(k_x) \propto \left[\frac{\sin \frac{k_x d}{2}}{\frac{k_x d}{2}} \right]^2 , \tag{6}$$

where $d = 5\mu$m.

The intensity pattern is thus the fourth power of the sinc. With $k_x d/2 = (2\pi/\lambda)(x/L)d/2 = (5\pi/100)x$. The FWMH of the intensity is twice the value of x for which $k_x d/2 = w$, where w is defined as the value of the argument of the sinc for which:

$$\left[\frac{\sin w}{w}\right]^4 = \frac{1}{2}. \tag{7}$$

$\sin w/w = 0.841$ for $w = 1$. That gives $x = 100/(5\pi)$ and the FWHM is $200/5\pi$ cm $= 12.7$ cm.

The zero order of the grating is at $\theta_x = 0$. The $+1$ and -1 orders are given by

$$\sin\theta = \pm\lambda/d = \pm 0.1 \tag{8}$$

The amplitude of the pattern for that angle is:

$$\left[\frac{\sin\frac{k\theta d}{2}}{\frac{k\theta d}{2}}\right]^4$$
$$\left[\frac{\sin 0.5\pi}{0.5\pi}\right]^4 = 0.16. \tag{9}$$

This grating has only 16% efficiency. The higher orders are much smaller, since they come even after the zero of the sinc.

3) Aperture diffraction

An aperture is made of an open rectangle of dimensions $h_1 \times W_1$. The central part of that rectangle has a rectangular obstruction of dimensions $h_2 \times W_2$. Find an expression for the far field diffraction pattern (intensity distribution). Sketch.

solution Using the superposition principle, the electric field distribution in Fraunhofer approximation is the difference in Fourier transform of two rectangular aperture

$$E(x', y') \propto \frac{1}{i\lambda z}(h_1 w_1)\text{sinc}(\frac{w_1 x'}{\lambda z})\text{sinc}(\frac{h_1 y'}{\lambda z}) - (h_2 w_2)\text{sinc}(\frac{w_2 x'}{\lambda z})\text{sinc}(\frac{h_2 y'}{\lambda z})) \qquad (10)$$

Which gives the intensity

$$\begin{aligned} I(x', y') &= \frac{1}{(\lambda z)^2}(h_1 w_1)^2\text{sinc}^2(\frac{w_1 x'}{\lambda z})\text{sinc}^2(\frac{h_1 y'}{\lambda z}) \\ &+ (h_2 w_2)^2\text{sinc}^2(\frac{w_2 x'}{\lambda z})\text{sinc}^2(\frac{h_2 y'}{\lambda z}) - 2(h_1 w_1)(h_2 w_2)\text{sinc}(\frac{w_1 x'}{\lambda z})\text{sinc}(\frac{h_1 y'}{\lambda z})\text{sinc}(\frac{w_2 x'}{\lambda z})\text{sinc}(\frac{h_2 y'}{\lambda z}) \end{aligned} \qquad (11)$$