Nonlinear Optics 2024 — Homework 3
Due Wednesday, February 21, 2024

1. Self-induced transparency

Self-induced transparency is one the first examples of soliton propagation in optics, based on coherent resonant propagation in an absorber. It applies mostly to inhomogeneously broadened systems, for which the (undamped) Bloch's interaction equations apply. The purpose of this problem is to derive the form of the steady state propagating pulse, the 2 π sech, in the simpler narrow line limit. The extension to the inhomogeneous broadening case adds a lot of mathematics but little of physics. It is essential when considering the evolution of a pulse towards steady state, but not necessary to derive the steady state. We therefore consider the system of Bloch's Maxwell equations in the slowly varying approximation, to be reduced to:

$$\dot{u} = (\omega_0 - \omega - \dot{\varphi})v \tag{1}$$

$$\dot{v} = -(\omega_0 - \omega - \dot{\varphi})u - \kappa\mathcal{E}w \tag{2}$$

$$\dot{w} = \kappa\mathcal{E}v \tag{3}$$

where the initial value for w at $t = -\infty$ is

$$w_0 = -pN_0. \tag{4}$$

p is the dipole moment of the transition, expectation value of the dipole moment operator $e\mathbf{r}$ between ground state and upper state of the two level transition $\langle 0|e\mathbf{r}|1\rangle$ In these equations, the light electric field is:

$$E(z,t) = \frac{1}{2}\tilde{\mathcal{E}}(z,t)e^{i(\omega t - kz)} = \frac{1}{2}\mathcal{E}(z,t)e^{i[\omega t + \varphi(z,t) - kz]}, \tag{5}$$

where $\tilde{\mathcal{E}}$ is the complex envelope, $\mathcal{E}$ the real amplitude, φ the phase, all slowly varying. ω is the average carrier frequency of the light, and ω_0 the transition frequency of the two-level system. The propagation equation, in terms of $\mathcal{E}$ and φ, in a retarded frame of reference (propagating at the group velocity of the host medium):

$$\frac{\partial \mathcal{E}(z,t)}{\partial z} = -\frac{\mu_0 \omega c}{2n} v(z,t) \tag{6}$$

$$\frac{\partial \varphi}{\partial z} = -\frac{\mu_0 \omega c}{2n} \frac{u(z,t)}{\mathcal{E}(z,t)}, \tag{7}$$

where we have explicitly written the (retarded) time t and space z dependence of the functions [dependence omitted in Eqs. (1), (2), (3) to simplify the notations].

The purpose of the problem is to find a shape preserving solution of the form:

$$\mathcal{E}(t) = \frac{a}{\tau_s}\text{sech}\left(\frac{t}{\tau_s} - bz\right), \tag{8}$$

and determine the parameters a and b.

We will for simplicity assume to be at exact resonance.

How does the resonance condition simplify the solution?

Guides to the self-preserving solution We have defined in class the tipping angle of the pseudo-polarization θ as:

$$\theta(z) = \int_{-\infty}^{\infty} \kappa \mathcal{E} dt \tag{9}$$

This "tipping angle" of the polarization a time integrated quantity, characteristic of the pulse at a particular position. We can define a time dependent quantity, which is the tipping angle of the polarization as the pseudo-polarization Bloch vector evolves under the influence of the electric field. To avoid confusion in notation, we will call this quantity β:

$$\beta = \int_{-\infty}^{t} \kappa \mathcal{E}(t') dt' \tag{10}$$

This function β will play a central role in finding the steady state solution of Bloch-Maxwell's equations. In order to find the steady state condition, all quantities will be expressed as a function of β. Note that in the retarded frame of reference that we have chosen, the 2π pulse is a shape preserving envelope at an envelope velocity V. Therefore, the key condition is:

$$\frac{\partial \mathcal{E}}{\partial z} = -\frac{1}{V} \frac{\partial \mathcal{E}}{\partial t}. \tag{11}$$

By substitution into Bloch's equation, derive the equation:

$$\frac{\partial^2 \beta}{\partial t^2} = \frac{1}{\tau^2} \sin \beta \tag{12}$$

and show that this equation corresponds to the equation of motion of an inverted pendulum.

Hint: Remember the geometric representation of Bloch's equations. Use the fact that the pseudo-polarization vector has a constant length $N_0 p$, and rotates by an angle $\beta(t)$. Express the coordinates v and w in terms of these quantities.

Show that the integration of Eq. (12) leads to:

$$\kappa \mathcal{E} = \frac{2}{\tau} \sin \frac{\beta}{2} \tag{13}$$

where

$$\tau = \sqrt{\frac{2n}{\mu_0 \omega c N_0 p \kappa V}} \tag{14}$$

The simplest approach is to start from the solution (13) and get to (12) by derivation.

Derive a differential equation for the quantity $q = \kappa \mathcal{E} \tau / 2$ (follow the same steps as above)

Integrate that equation to find the 2 π sech pulse.

Properties of sech $\text{sech}^2 x = 1 - \tanh^2 x$

$\frac{d\text{sech} x}{dx} = -\tanh \text{sech} x$

$\frac{d\text{sech}^{-1} x}{dx} = -\frac{i}{x\sqrt{1-x^2}}$