# Nonlinear Optics 2023 — Homework 5

Due Monday, April 5, 2023

**Gaussian beams with phase conjugated mirror**

Reminder: equation of a Gaussian beam:

$$\tilde{\mathcal{E}}(x,y,z) = \frac{\mathcal{E}_0}{\sqrt{1+z^2/\rho_0^2}} e^{-i\Theta} \times e^{-ik_\ell(r^2)/2R} \times e^{-(r^2)/w^2} \tag{1}$$

where the various parameters are:

$$\begin{aligned}
r^2 &= x^2+y^2 \\
R = R(z) &= z+\rho_0^2/z \\
w = w(z) &= w_0\sqrt{1+z^2/\rho_0^2} \\
\Theta = \Theta(z) &= \arctan(z/\rho_0) \\
\rho_0 &= \rho(z=0) = \frac{n\pi w_0^2}{\lambda}. \\
\rho &= \frac{n\pi w^2}{\lambda}
\end{aligned}$$

A flat, ideal phase conjugate mirror transforms Eq. (1) into:

$$\tilde{\mathcal{E}}(x,y,z) = \frac{\mathcal{E}_0}{\sqrt{1+z^2/\rho_0^2}} e^{i\Theta} \times e^{ik_\ell(r^2)/2R} \times e^{-(r^2)/w^2} \tag{2}$$

In terms of complex q-parameter, Eq. (1) is:

$$\tilde{\mathcal{E}}(x,y,z) = \frac{\mathcal{E}_0}{\sqrt{1+z^2/\rho_0^2}} e^{-i\Theta} \times e^{-ik_\ell(r^2)/2\tilde{q}} \tag{3}$$

## 1.Find the q-parameter transformation for a flat phase conjugated mirror

In terms of complex q-parameter, the complex conjugate of Eq. (1) is:

$$\tilde{\mathcal{E}}(x,y,z) = \frac{\mathcal{E}_0}{\sqrt{1+z^2/\rho_0^2}} e^{i\Theta} \times e^{ik_\ell(r^2)/2R} \times e^{-(r^2)/w^2}, \tag{4}$$

which is obtained by making $q_2 = -q_1^*$ in Eq. (3).

$$\frac{1}{\tilde{q}} = \frac{1}{R} - \frac{i}{\rho}$$

After phase conjugation:

$$\frac{1}{\tilde{q}} = -\frac{1}{R} - \frac{i}{\rho}$$

## 2. Write this transformation in terms of ABCD matrix

Without phase conjugation:

$$\frac{1}{q_2} = \frac{C + \frac{D}{q_1}}{A + \frac{B}{q_1}} \tag{5}$$

Hint: with phase conjugation the expression may involve complex conjugation.

$$\frac{1}{q_2} = \frac{C + \frac{D}{q_1^*}}{A + \frac{B}{q_1^*}} \tag{6}$$

The ABCD matrix of a flat phase conjugated mirror is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

It could also be:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

## 3. Stability condition of a simple cavity

Having determined the ABCD matrix for a phase congution mirror, apply this result to analyze the satbility of a simple cavity with a curved (normal) mirror and a flat phase conjugated mirror.
Hint: make 2 round-trips.
Find the stabiliity condition and the beam size.
The ABCD matrix of the cavity for one round-trip is:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & -1 \end{pmatrix}$$

Note that it is independent of the cavity length $L$.
For 2 round-rips:

$$\begin{pmatrix} 1 & 0 \\ -\frac{2}{f} & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & -1 \end{pmatrix}$$

Stability: $\frac{A+D}{2} = 0$
Rayleigh parameter and beam size: $\frac{0}{0}$ - undetermined - can be anything.

## 4. Why the hint in the previous question?

Because of the complex conjugation, the $q$ parameter does not repeat after 1 round-trip.

$$\frac{1}{\tilde{q}_2} = -\frac{1}{f} - \frac{1}{q_1^*}$$

But after 2:

$$\frac{1}{\tilde{q}_3} = -\frac{1}{f} - \frac{1}{q_2^*} = -\frac{2}{f} - \frac{1}{q_1}$$

## 5. Non degenerate FWM: find the resonance condition for longitudinal modes

The process is:

$$\omega_p + \omega_p - \omega_1 - \omega_2 = 0. \quad (9)$$

The resonance condition is $kL = 2N\pi = K_1 L + k_2 L = \varphi_1 + \varphi_2$.

$$\begin{aligned} \varphi_1 &= \frac{\omega_1}{c} L \\ \varphi_1 &= \frac{\omega_2}{c} L = \frac{2\omega_p - \omega_1}{c} L \end{aligned}$$

$$\varphi = \varphi_1 + \varphi_2 = \frac{2\omega_p}{c} L = 2N\pi$$

Only selected PUMP frequencies resonate with the cavity! '