Phase Shifts upon Transmission and Reflection

Most often, phase shifts at interfaces are a simple consequence of energy conservation. Conversely, the phase shift properties in simple devices can be used to determine the direction of the flow of energy. A few simple examples are given here.

0.1 The symmetrical interface

Let us consider first the very simple situation sketched in Fig. 1. The interface can be a mirror with a reflecting coating on the front face and an antireflection coating on the back face. We are only interested in fields propagating *outside* the mirror. The energy conservation relation between the reflected (field reflection coefficient $\tilde{r}$) and transmitted (field transmission coefficient $\tilde{t}$) waves implies:

$$|\tilde{r}|^2 + |\tilde{t}|^2 = 1, \tag{1}$$

where we assumed a unity field amplitude.

Another relation can be found by adding another incident field of amplitude 1 (beam 2 in the figure), and taking advantage of the symmetry. Summing the intensities:

$$|\tilde{r} + \tilde{t}|^2 + |\tilde{r} + \tilde{t}|^2 = 2. \tag{2}$$

Combination of Eqs. (1) and (2) leads to

$$2[\tilde{r}\tilde{t}^* + \tilde{r}^*\tilde{t}] = 0, \tag{3}$$

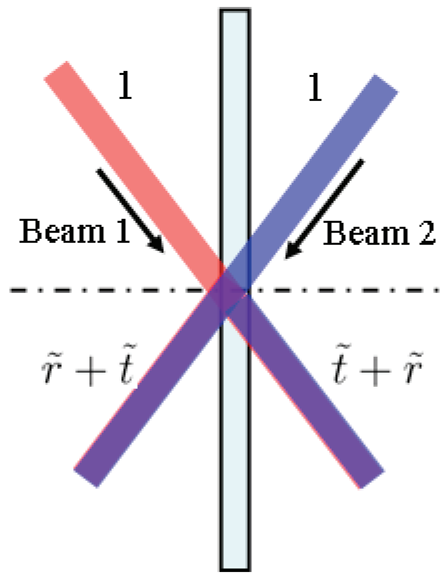

Figure 1: Reflection and transmission by an interface between two identical media

which implies that the phase shifts upon transmission and reflection are complementary:

$$\varphi_r - \varphi_t = \frac{\pi}{2}. \tag{4}$$

It is because of the latter phase relation that the antiresonant ring reflects back all the incident radiation, and has zero losses if $|\tilde{r}|^2 = |\tilde{t}|^2 = 0.5$.

0.2 Coated interface between two different dielectrics

Let us consider – as in Fig. 2 – a partially reflecting coating at an interface between air (index 1) and a medium of index n. A light beam of amplitude $\mathcal{E}_1 = 1/\sqrt{\cos\theta_1}$ is incident from the air, at an angle of incidence θ_1. The transmitted beam is refracted at the angle θ_2, and has an amplitude $\tilde{t}_1/\sqrt{cos\theta_1}$. The reflected beam has an amplitude $\tilde{r}_1/\sqrt{cos\theta_1}$. We take the vertical (orthogonal to the figure) dimension of the beam to be unity, as well as the distance covered by the beam on the interface in the plane of the figure. To calculate energy conservation, we compare the products $n_i|\tilde{\mathcal{E}}|^2 A$ where $n_i = 1$ left of the interface, $n_i = n$ right of the interface, and $A = 1 \times \cos\theta$. As in the previous section, we will be considering a similar beam incident from the right, with an amplitude $\mathcal{E}_2 = 1/\sqrt{n\cos\theta_2}$ incident at an angle θ_2 on the dielectric/air interface. The choice of these incident electric field amplitudes is such that the same “energy” products $n_i|\tilde{\mathcal{E}}|^2 A = A$ apply on both sides of the interface, above the dash-dotted line in Fig. 2.

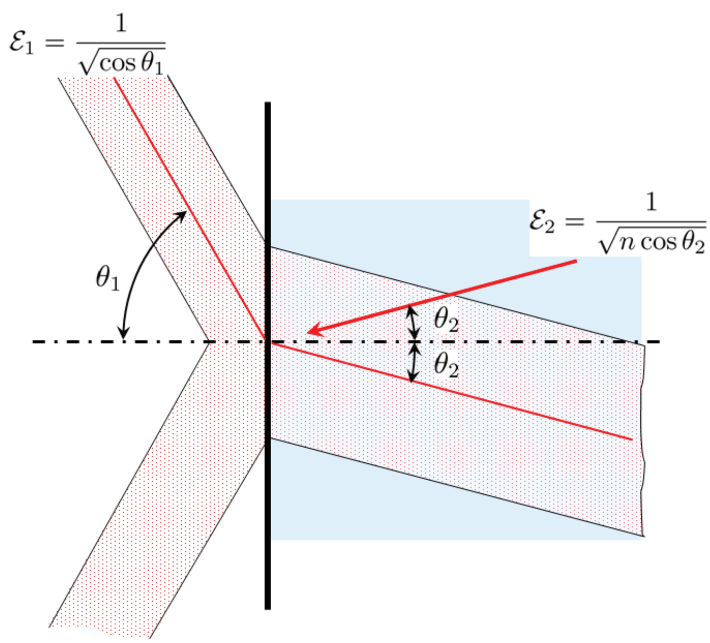

Figure 2: Reflection and transmission by an interface between air and a dielectric.

Energy conservation leads to the relation:

$$|\tilde{r}_1|^2 + |\tilde{t}_1|^2 \frac{n\cos\theta_2}{\cos\theta_1} = 1, \tag{5}$$

where we took into account the change in beam cross section upon refraction. We have a similar energy conservation equation for a beam of amplitude $\mathcal{E}_2 = 1/\sqrt{n\cos\theta_2}$ incident at an angle θ_2 on the dielectric/air interface:

$$|\tilde{r}_2|^2 + |\tilde{t}_2|^2 \frac{\cos\theta_1}{n\cos\theta_2} = 1. \tag{6}$$

From Eqs. (5) and (6) we get directly the relation:

$$|t_1|^2 \cdot |t_2|^2 = T_1 T_2 = (1-|r_1|^2)(1-|r_2|^2) = (1-R_1)(1-R_2) \tag{7}$$

The amplitude of the reflection coefficient is equal on both sides of the interface. For the phase, the only sign relation consistent with energy conservation in a Gires-Tournois interferometer, and with the known phase shift on pure dielectric interfaces, is:

$$\tilde{r}_1 = -\tilde{r}_2^*, \tag{8}$$

or, $r_1 = r_2$, with the relation between phase angles:

$$\boxed{\varphi_{r,1} = -\varphi_{r,2} - \pi.} \tag{9}$$

Since $|\tilde{r}_1|^2 = |\tilde{r}_2|^2$, Eqs (5) and (6) lead to:

$$\boxed{|t_1|\sqrt{\frac{n\cos\theta_2}{\cos\theta_1}} = |t_2|\sqrt{\frac{\cos\theta_1}{n\cos\theta_2}}.} \tag{10}$$

The amplitudes of the transmission coefficients are not equal, but in the ratio $|t_2|/|t_1| = n\cos\theta_2/\cos\theta_1$, a relation that satisfies Fresnel equations, and results simply from energy conservation.

In order to find a relation between the phase shift upon transmission and reflection, we consider the energy conservation for light incident from the upper half of the figure (the axis of symmetry being the dashed normal to the interface):

$$\begin{aligned} 1+1 = \cos\theta_1 \left|\frac{\tilde{r}_1}{\sqrt{\cos\theta_1}} + \frac{\tilde{t}_2}{\sqrt{n\cos\theta_2}}\right|^2 \\ + n\cos\theta_2 \left|\frac{\tilde{r}_2}{\sqrt{n\cos\theta_2}} + \frac{\tilde{t}_1}{\sqrt{\cos\theta_1}}\right|^2. \end{aligned} \tag{11}$$

Expanding:

$$\begin{aligned} 2 = |r_1|^2 + |r_2|^2 + |t_2|^2\frac{\cos\theta_1}{n\cos\theta_2} + |t_1|^2\frac{n\cos\theta_2}{\cos\theta_2} \\ +(\tilde{r}_1\tilde{t}_2^* + \tilde{r}_1^*\tilde{t}_2)\sqrt{\frac{\cos\theta_1}{n\cos\theta_2}} + (\tilde{r}_2\tilde{t}_1^* + \tilde{r}_2^*\tilde{t}_1)\sqrt{\frac{n\cos\theta_2}{\cos\theta_1}} \end{aligned} \tag{12}$$

Taking into account the energy conservation relations (5) and (6), leads to:

$$(\tilde{r}_1\tilde{t}_2^* + \tilde{r}_1^*\tilde{t}_2)\cos\theta_1 + (\tilde{r}_2\tilde{t}_1^* + \tilde{r}_2^*\tilde{t}_1)n\cos\theta_2 = o. \tag{13}$$

We can re-write Eq. (13)

$$2|r_1||t_2|\left\{\cos(\varphi_{r,1} - \varphi_{t,2})\right\}\cos\theta_1 = -2|r_2||t_1|\left\{\cos(\varphi_{r,2} - \varphi_{t,1})\right\}n\cos\theta_2. \tag{14}$$

Equation 13 leads also to the following trigonometric relations between phase shifts upon transmission and reflection:

$$\frac{\cos(\varphi_{r,1} - \varphi_{t,2})}{\cos(\varphi_{r,2} - \varphi_{t,1})} = -1, \tag{15}$$

which leads to the relation between phase angles:

$$\varphi_{t,2} - \varphi_{r,1} = \varphi_{r,2} - \varphi_{t,1} + (2n+1)\pi. \tag{16}$$

or

$$\varphi_{t,1} + \varphi_{t,2} = \varphi_{r,1} + \varphi_{r,2} + (2n+1)\pi. \tag{17}$$

Taking into account the relation Eq. (9) between phase angled in reflection:

$$\boxed{\varphi_{t,1} + \varphi_{t,2} = 2n\pi.} \tag{18}$$

We can thus conclude that the transmission from the two sides of the interface have complementary phase angle. We do not find any relation between the phase in reflection and the phase in transmission. The phases in reflection on either side of the interface are related by Eq. (9). A final relation that is consistent with the above and very useful in the calculation of the transmission of a Fabry-Perot is:

$$\boxed{\tilde{t}_1\tilde{t}_2 = 1 + \tilde{r}_1\tilde{r}_2 = 1 - |r|^2 = T = 1 - R.} \tag{19}$$

Combining Eqs (18) and Eq. (19) leads us to conclude that $\tilde{t}_1 = -\tilde{t}_2$. Multiplying Eq. (19) by its complex conjugate is consistent with the energy conservation Eq. (7):

$$\begin{aligned} T_1 \cdot T_2 = [\tilde{t}_1\tilde{t}_2] \cdot [\tilde{t}_1\tilde{t}_2]^* &= [1 + \tilde{r}_1\tilde{r}_2] \cdot [1 + \tilde{r}_1\tilde{r}_2]^* \\ &= 1 + \tilde{r}_1\tilde{r}_1^*\tilde{r}_2\tilde{r}_2^* + \tilde{r}_1\tilde{r}_2 + \tilde{r}_1^*\tilde{r}_2^* \\ &= 1 + R_1R_2 - R_1 - R_2 = (1 - R_1)(1 - R_2) \end{aligned} \tag{20}$$

where again we took into account the relation $r_1 = -r_2^*$. Finally, it can be verified that Fresnel equations are satisfied by Eq. (19) for both polarizations. Note however that Fresnel relations apply to a *single interface* where the phase shift in transmission is zero, and the phase shift on reflection 0 or π. The situation is different for a multilayer coating deposited on a dielectric surface. It is somewhat surprising that there is no relation implied through energy conservation between φ_r and φ_t.