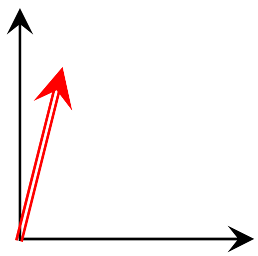
HALF WAVE: it is a phase shift of π $(\lambda/2)$

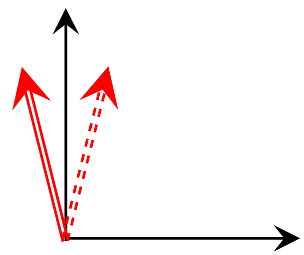
$E_{x0}\cos\omega t$

$E_{y0}\cos\omega t$

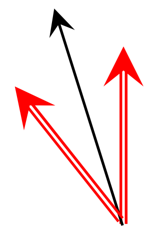


$E_{x0}\cos(\omega t + \pi)$

$E_{y0}\cos\omega t$



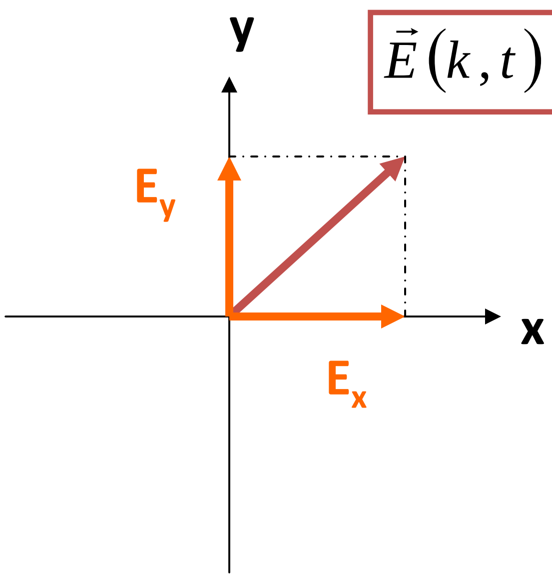
with respect to polarization, the angle is doubled



Matrix treatment of polarization

- Consider a light ray with an instantaneous E-vector as shown

$$\vec{E}(k,t) = \hat{i}E_x(k,t) + \hat{j}E_y(k,t)$$



Matrix treatment of polarization

- Combining the components

$$\vec{E} = \hat{i}E_{ox}e^{i(\omega t - kz + \varphi_x)} + \hat{j}E_{oy}e^{i(\omega t - kz + \varphi_y)}$$

$$\vec{E} = \left[\hat{i}E_{ox}e^{i\varphi_x} + \hat{j}E_{oy}e^{i\varphi_y}\right]e^{i(\omega t - kz)}$$

$$\vec{E} = \tilde{E}_o e^{i(\omega t - kz)}$$

- The terms in brackets represents the complex amplitude of the plane wave

Jones Vectors

- The state of polarization of light is determined by
  - the relative amplitudes (E_{ox}, E_{oy}) and,
  - the relative phases ($\delta = \varphi_y - \varphi_x$)

  of these components
- The complex amplitude is written as a two-element matrix, the Jones vector

$$\tilde{E}_o = \begin{bmatrix} \tilde{E}_{ox} \\ \tilde{E}_{oy} \end{bmatrix} = \begin{bmatrix} E_{ox} e^{i\varphi_x} \\ E_{oy} e^{i\varphi y} \end{bmatrix} = e^{i\varphi_x} \begin{bmatrix} E_{ox} \\ E_{oy} e^{i\delta} \end{bmatrix}$$

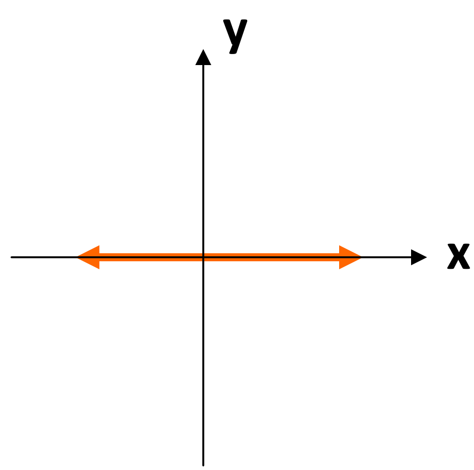
Jones vector: Horizontally polarized light

The arrows indicate the sense of movement as the beam approaches you

- The electric field oscillations are only along the x-axis
- The Jones vector is then written,

$$\tilde{E}_o = \begin{bmatrix} \tilde{E}_{ox} \\ \tilde{E}_{oy} \end{bmatrix} = \begin{bmatrix} E_{ox} e^{i\varphi_x} \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where we have set the phase φ_x = 0, for convenience



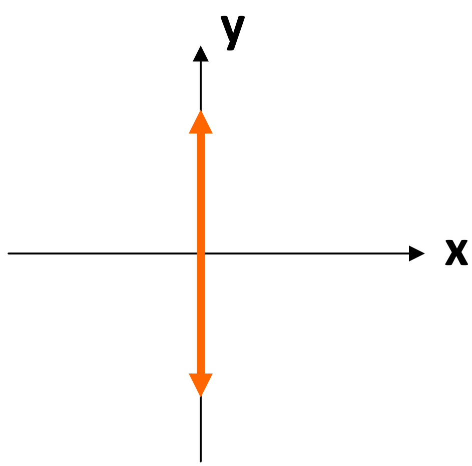
The normalized form is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Jones vector: Vertically polarized light

- The electric field oscillations are only along the y-axis
- The Jones vector is then written,

$$\tilde{E}_o = \begin{bmatrix} \tilde{E}_{ox} \\ \tilde{E}_{oy} \end{bmatrix} = \begin{bmatrix} 0 \\ E_{oy} e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} 0 \\ A \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Where we have set the phase $\varphi_y = 0$, for convenience

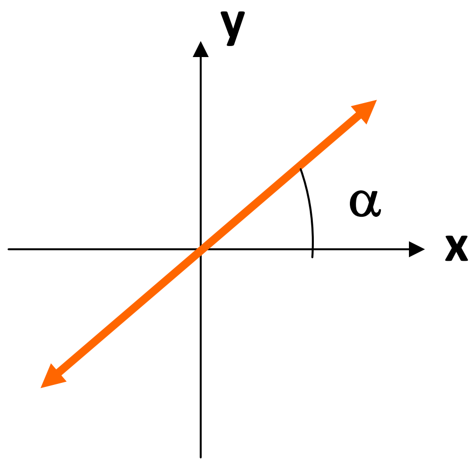


The normalized form is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Jones vector: Linearly polarized light at an arbitrary angle

- If the phases are such that $\delta = m\pi$ for
  $m = 0, \pm1, \pm2, \pm3, \ldots$
- Then we must have,



$$\frac{E_x}{E_y} = (-1)^m \frac{E_{ox}}{E_{oy}}$$

and the Jones vector is simply a line inclined at an angle $\alpha = \tan^{-1}(E_{oy}/E_{ox})$

since we can write

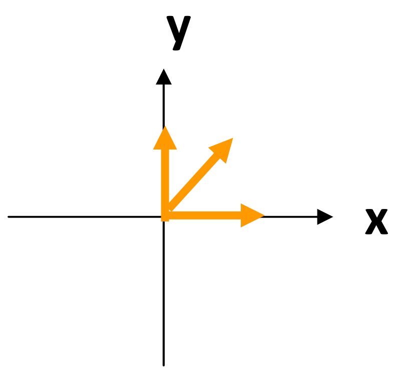
$$\tilde{E}_o = \begin{bmatrix} \tilde{E}_{ox} \\ \tilde{E}_{oy} \end{bmatrix} = A(-1)^m \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

The normalized form is

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

Circular polarization

- Suppose $E_{ox} = E_{oy} = A$ and E_x leads E_y by $90^o = \pi/2$
- At the instant E_x reaches its maximum displacement (+A), E_y is zero
- A fourth of a period later, E_x is zero and $E_y = +A$



$t=0$, $E_y = 0$, $E_x = +A$

$t=T/8$, $E_y = +A\sin 45^o$, $E_x = A\cos 45^o$

$t=T/4$, $E_y = +A$, $E_x = 0$

Circular polarization

- For these cases it is necessary to make $\varphi_y < \varphi_x$. Why?
- This is because we have chosen our phase such that the time dependent term (ωt) is positive

$$E_x = E_{ox} e^{i(\omega t - kz + \varphi_x)}$$

$$E_y = E_{oy} e^{i(\omega t - kz + \varphi_y)}$$

Circular polarization

- In order to clarify this, consider the wave at z=0
- Choose φ_x=0 and φ_y=-ε, so that $\varphi_x > \varphi_y$
- Then our E-fields are

$$E_x = Ae^{i(\omega t)}$$

$$E_y = Ae^{i(\omega t - \varepsilon)}$$

- The negative sign before ε indicates a lag in the y-vibration, relative to x-vibration

Circular polarization

- To see this lag (in action), take the real parts
- We can then write

$$E_x = A\cos\omega t$$

$$E_y = A\cos\left(\omega t - \frac{\pi}{2}\right) = A\sin\omega t$$

- Remembering that $\omega=2\pi/T$, the path travelled by the e-vector is easily derived
- Also, since $E^2=E_x^2 + E_y^2 = A^2(\cos^2\omega t + \sin^2\omega t)= A^2$
- The tip of the arrow traces out a circle of radius A.

A= 1 for normalization.

Circular polarization

- The Jones vector for this case – where E_x leads E_y is

$$\tilde{E}_o = \begin{bmatrix} E_{ox} e^{i\varphi_x} \\ E_{oy} e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ A e^{i\pi/2} \end{bmatrix} = A \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- The normalized form is,

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- This vector represents circularly polarized light, where E rotates counterclockwise, viewed head-on
- This mode is called left-circularly polarized light
- What is the corresponding vector for right-circularly polarized light?

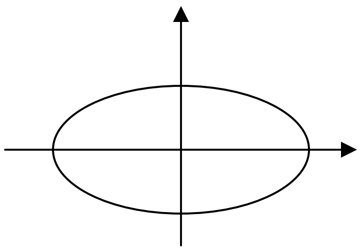
Replace π/2 with -π/2 to get $$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Elliptically polarized light

- If $E_{ox} \neq E_{oy}$, e.g. if E_{ox}=A and E_{oy} = B
- The Jones vector can be written

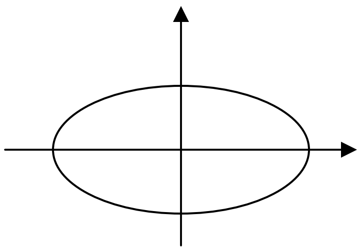
$$\begin{bmatrix} A \\ iB \end{bmatrix}$$

Type of rotation?



$$\begin{bmatrix} A \\ -iB \end{bmatrix}$$

Type of rotation?



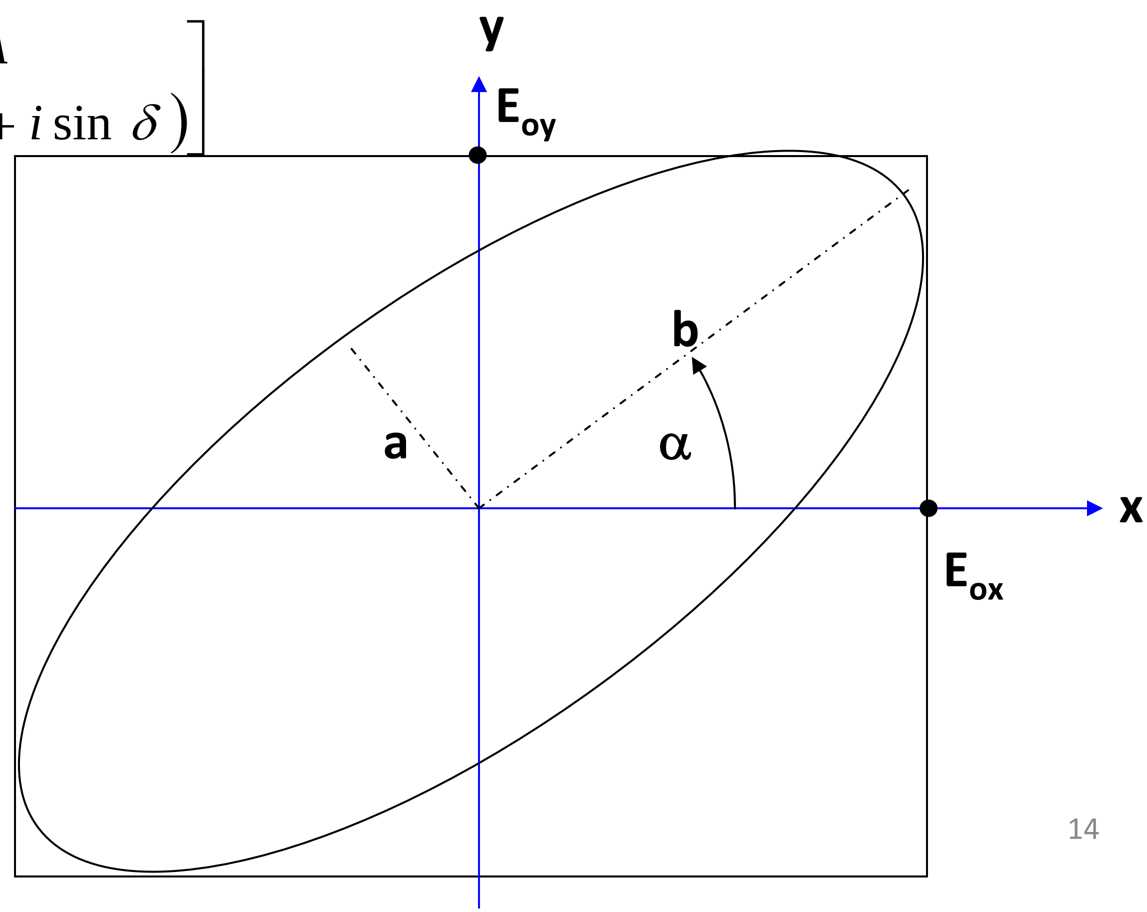
What determines the major or minor axes of the ellipse?

Jones vector and polarization

- In general, the Jones vector for the arbitrary case is an ellipse (δ≠ mπ; δ≠(m+1/2)π)

$$\tilde{E}_o = \begin{bmatrix} E_{ox} \\ E_{oy} e^{i\delta} \end{bmatrix} = \begin{bmatrix} A \\ B(\cos\delta + i\sin\delta) \end{bmatrix}$$

$$\tan 2\alpha = \frac{2E_{ox}E_{oy}\cos\delta}{E_{ox}^2 - E_{oy}^2}$$



Summary:
Given the expression for the field:

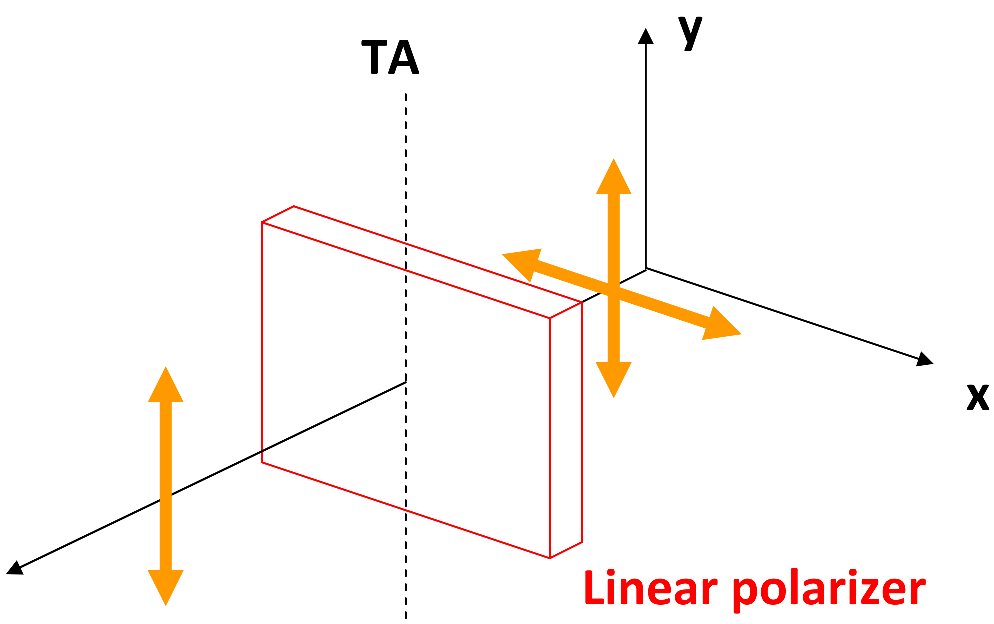
$$E = \left(E_{0x}e^{i\varphi_x}\hat{i} + E_{0y}e^{i\varphi_y}\hat{j}\right)e^{i(\omega t - kz)}$$

the components of the Jones matrix are:

$$\begin{vmatrix} \dfrac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}} \\ \dfrac{E_{0y}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i(\varphi_y - \varphi_x)} \end{vmatrix}$$

Optical elements: Linear polarizer

- Selectively removes all or most of the E-vibrations except in a given direction



Jones matrix for a linear polarizer

Consider a linear polarizer with transmission axis along the vertical (y). Let a 2X2 matrix represent the polarizer operating on vertically polarized light.
The transmitted light must also be vertically polarized. Thus,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Operating on horizontally polarized light,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $$M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear polarizer with TA vertical.

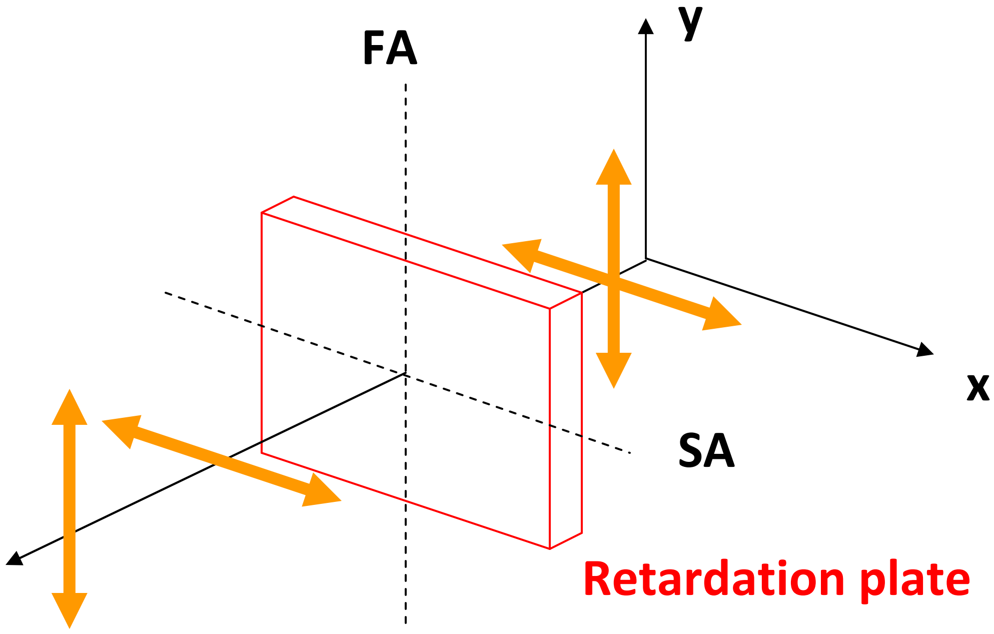
Jones matrix for a linear polarizer

- For a linear polarizer with a transmission axis at θ

$$M = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Optical elements: Phase retarder

- Introduces a phase difference ($\Delta\varphi$) between orthogonal components
- The fast axis (FA) and slow axis (SA) are shown



Jones matrix of a phase retarder

- We wish to find a matrix which will transform the elements as follows:

$$E_{ox}e^{i\varphi_x} \quad \text{into} \quad E_{ox}e^{i(\varphi_x+\varepsilon_x)}$$

$$E_{oy}e^{i\varphi_y} \quad \text{into} \quad E_{oy}e^{i(\varphi_y+\varepsilon_y)}$$

- It is easy to show by inspection that,

$$M = \begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix}$$

- Here ε_x and ε_y represent the advance in phase of the components

Jones matrix of a Quarter Wave Plate

- Consider a quarter wave plate for which $|\Delta\varepsilon| = \pi/2$
- For $\varepsilon_y - \varepsilon_x = \pi/2$ (Slow axis vertical)
- Let $\varepsilon_x = -\pi/4$ and $\varepsilon_y = \pi/4$
- The matrix representing a Quarter wave plate, with its slow axis vertical is,

$$M = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{-i\pi/4} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Jones matrices: HWP

- For |Δε| = π

$$M = \begin{bmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

HWP, SA vertical

$$M = \begin{bmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix} = e^{i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

HWP, SA horizontal

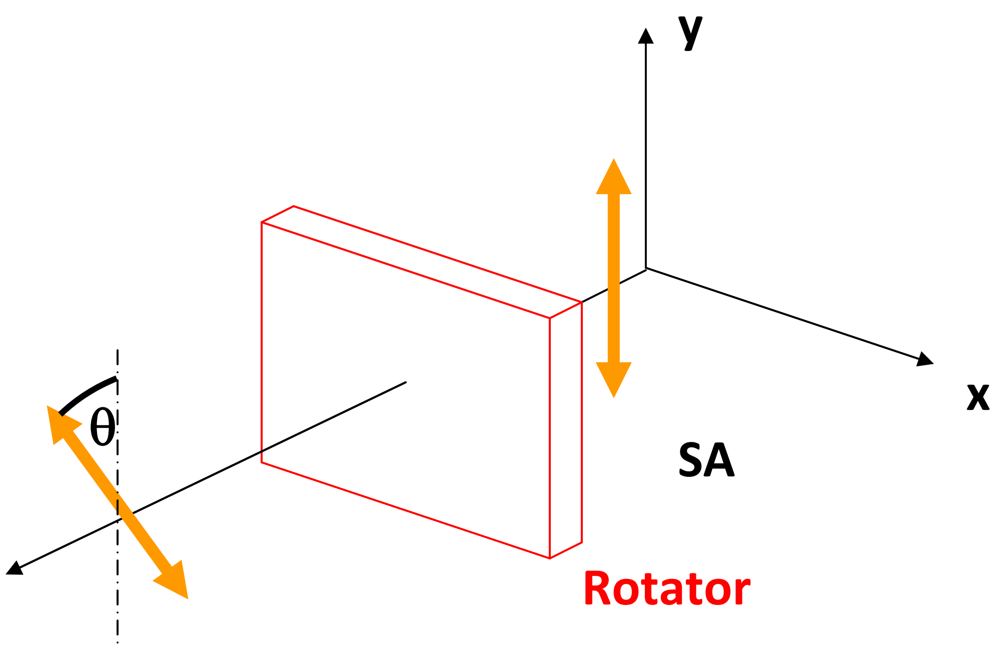
Jones matrix of a Quarter Wave Plate

- Consider a quarter wave plate for which $|\Delta\varepsilon| = \pi/2$
- For $\varepsilon_y - \varepsilon_x = \pi/2$ (Slow axis vertical)
- Let $\varepsilon_x = -\pi/4$ and $\varepsilon_y = \pi/4$
- The matrix representing a Quarter wave plate, with its slow axis vertical is,

$$M = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{-i\pi/4} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Optical elements: Rotator

- Rotates the direction of linearly polarized light by a particular angle θ

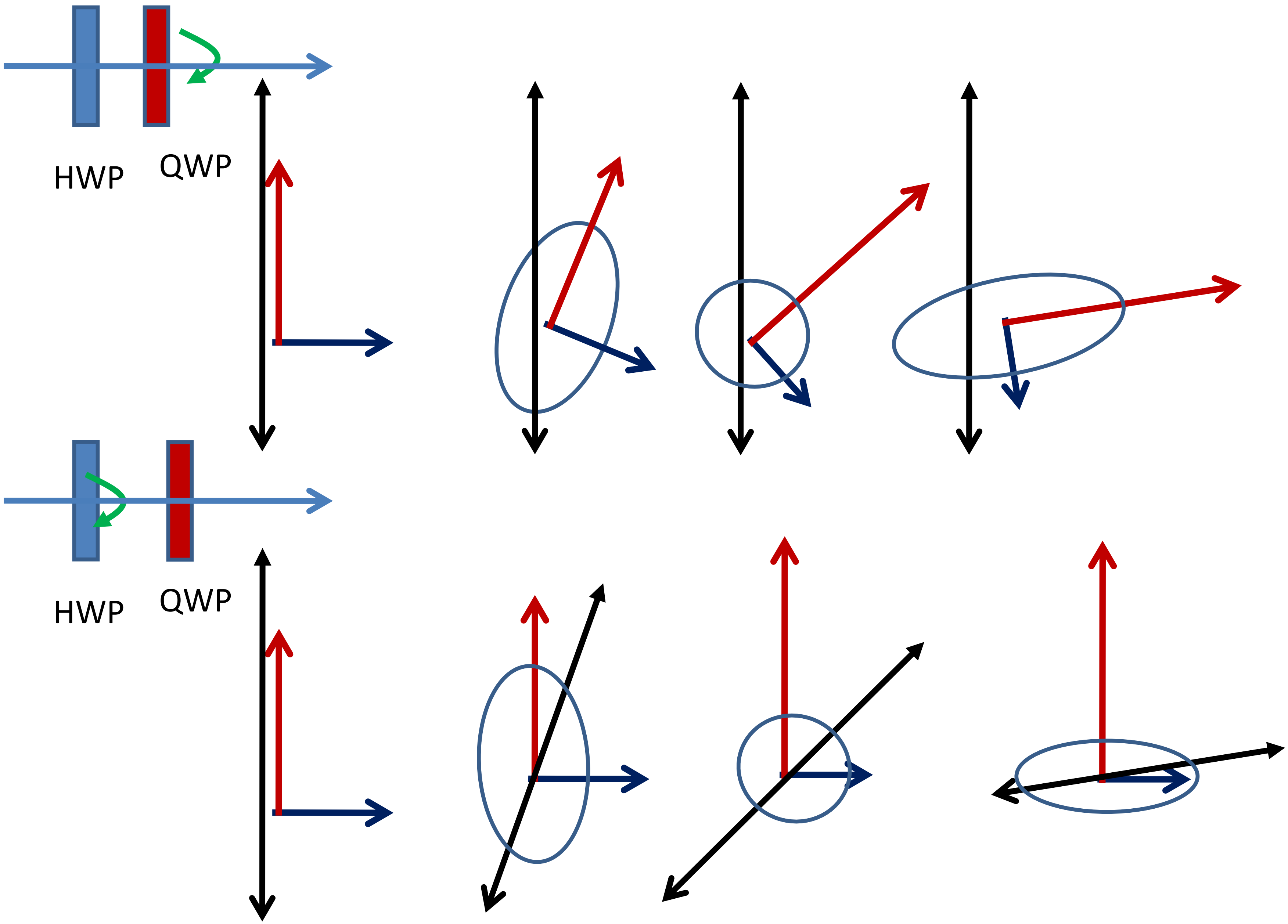


Jones matrix for a rotator

- An E-vector oscillating linearly at θ is rotated by an angle β
- Thus, the light must be converted to one that oscillates linearly at (β + θ)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos(\beta+\theta) \\ \sin(\beta+\theta) \end{bmatrix}$$

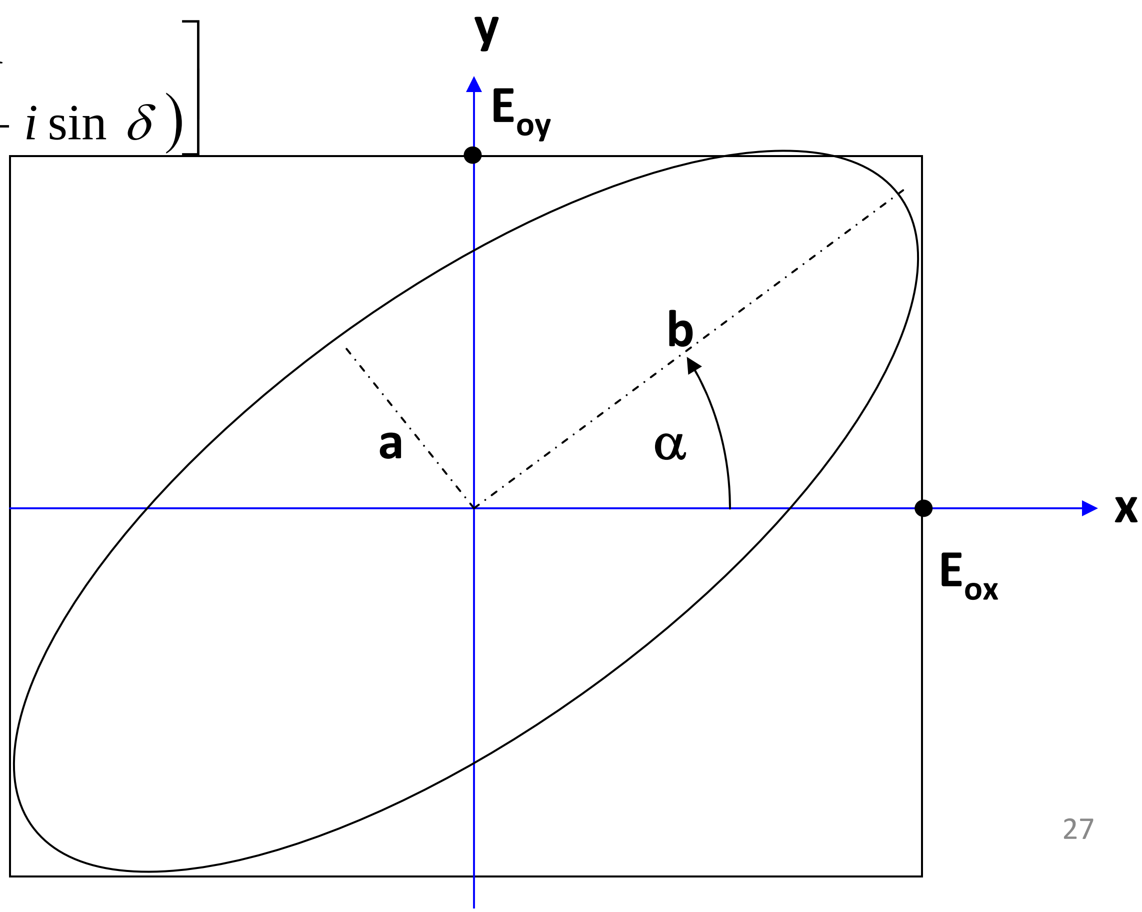
- One then finds $M = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$


HWP
QWP
HWP
QWP

Jones vector and polarization

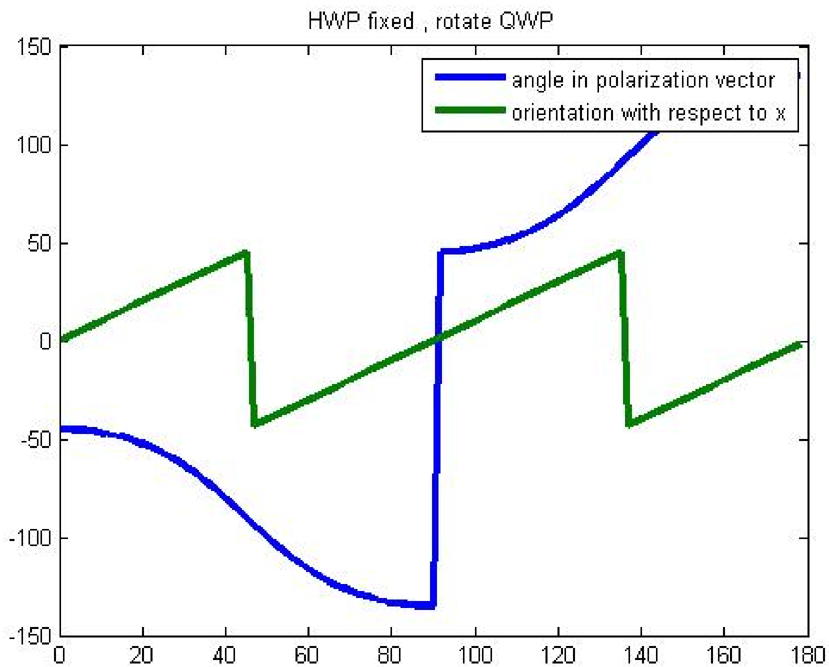
- In general, the Jones vector for the arbitrary case is an ellipse ($\delta \neq m\pi$; $\delta \neq (m+1/2)\pi$)

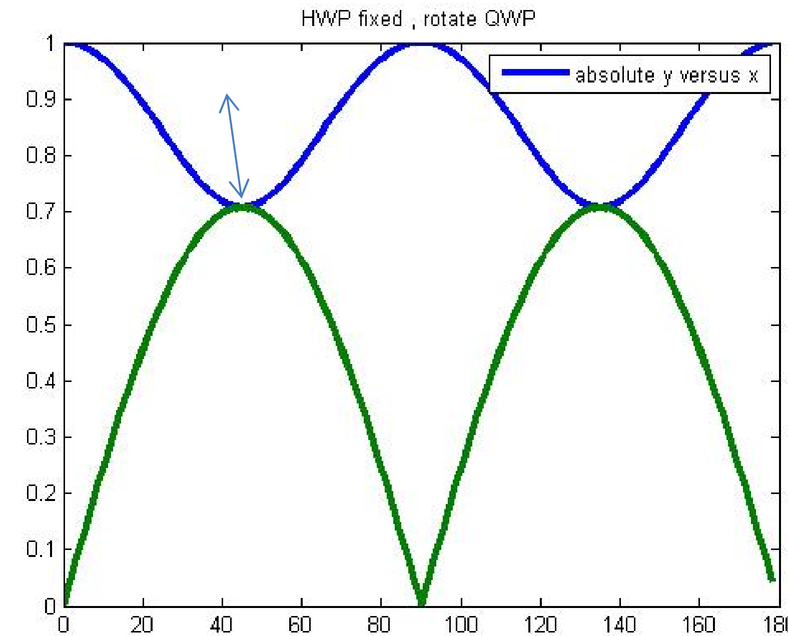
$$\tilde{E}_o = \begin{bmatrix} E_{ox} \\ E_{oy} e^{i\delta} \end{bmatrix} = \begin{bmatrix} A \\ B(\cos\delta + i\sin\delta) \end{bmatrix}$$

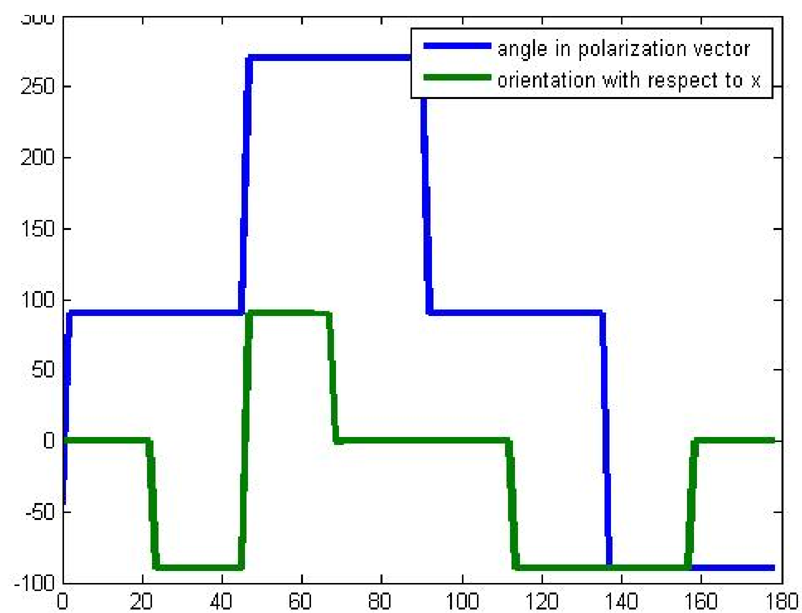


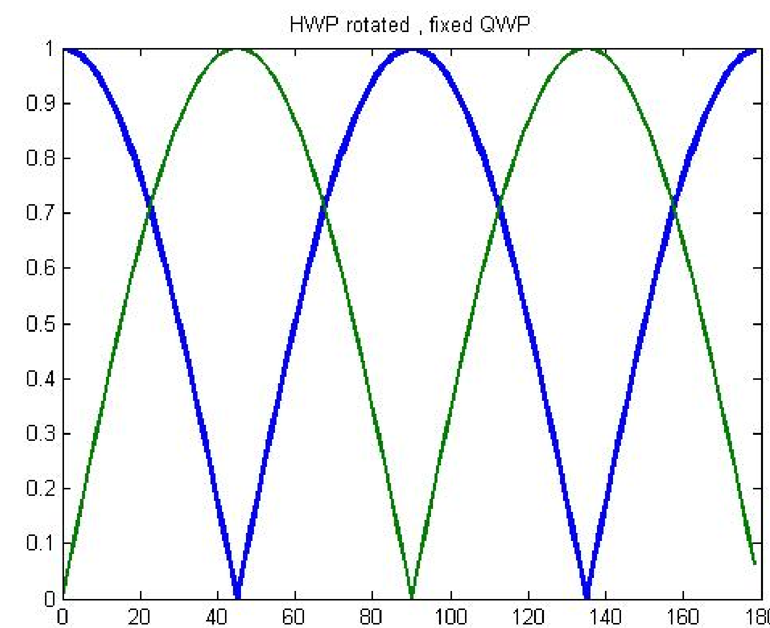
Problems

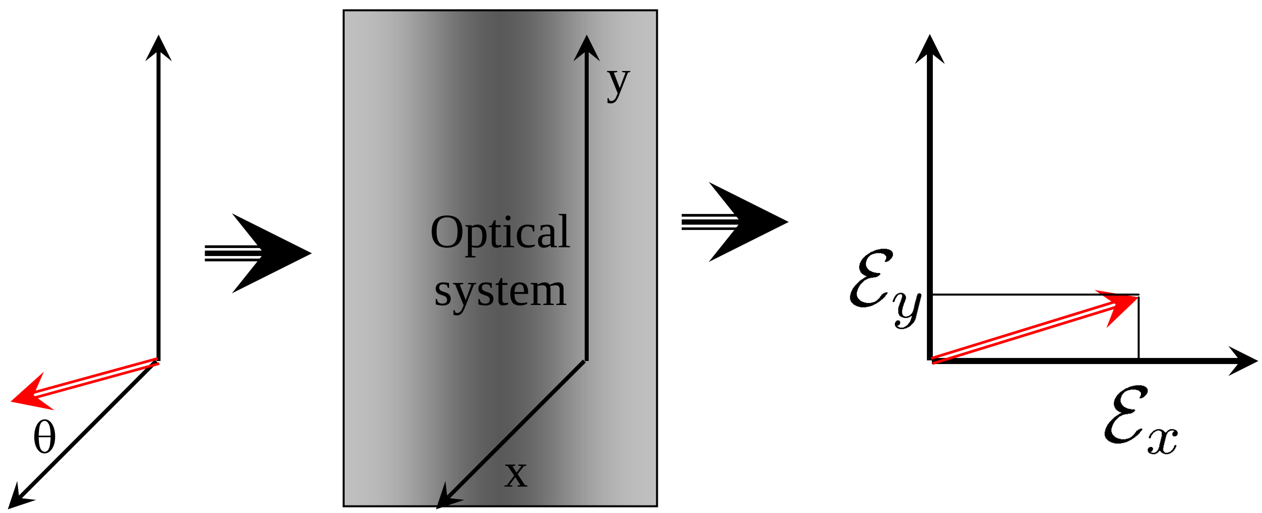
$$\tan 2\alpha = \frac{2E_{ox}E_{oy}\cos\delta}{E_{ox}^2 - E_{oy}^2}$$

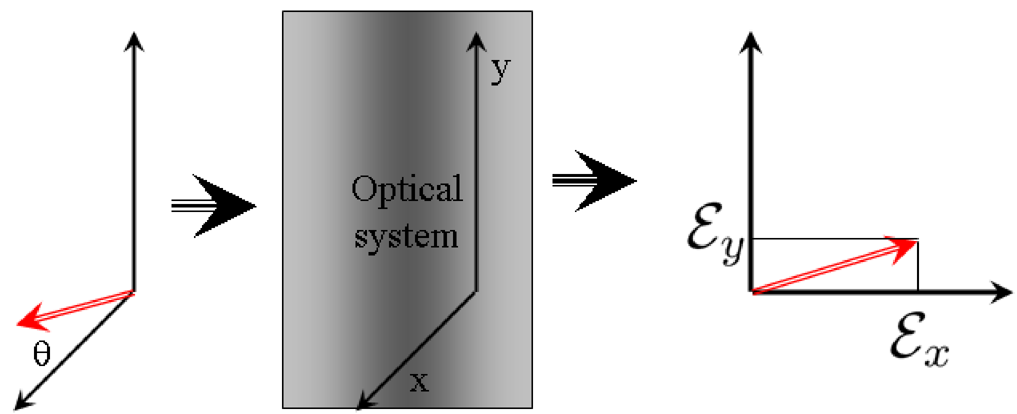

HWP fixed , rotate QWP
angle in polarization vector
orientation with respect to x


HWP fixed , rotate QWP
absolute y versus x


angle in polarization vector
orientation with respect to x


HWP rotated , fixed QWP


θ
Optical
system
y
x
$\mathcal{E}_y$
$\mathcal{E}_x$



θ
Optical
system
y
x
$\mathcal{E}_y$
$\mathcal{E}_x$

Spectral interferometry