Optics 463 — Homework 7 - Gaussian beams (Solutions)
due Tuesday, November 6, 2018

1 Plano-convex lens

Consider the plano-convex lens sketched in Fig. 1. The lens index is $n = 2$. Its thickness is $d = 1$ cm, and the radius of curvature of the curved surface is $R = 1$ cm. A collimated beam is sent through the plane face. Find the focal distance x using geometrical optics.

Next consider the beam to be a Gaussian of $w_0 = 100\mu$m incident on the plane surface, wavelength 1 μm. Find the location x of the beam waist (don't bother calculating the size).

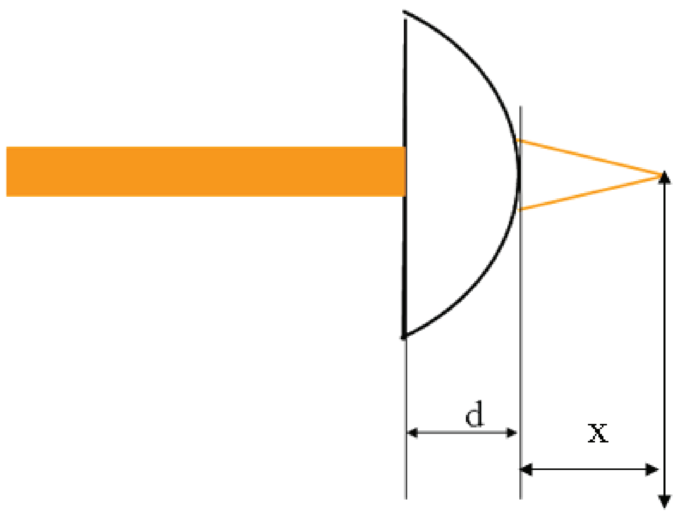

Figure 1:

Solution

Geometrical

The ABCD matrix equation for the entire system is:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{n_2-n_1}{n_1(-R)} & \frac{n_2}{n_1} \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

Plugging in the values for the constants and multiplying just the matrices for the lens (flat entrance face to curved output face) gives:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}. \tag{2}$$

Check: the determinant of the ABCD matrix is 1 since the beam begins and ends in the same medium. Multiplying in the final propagation matrix leads to:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1-x & \frac{1+x}{2} \\ -1 & \frac{1}{2} \end{pmatrix}. \tag{3}$$

The ray transfer equations is:

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} y \\ \alpha \end{pmatrix} \tag{4}$$

The focus is defined as where $y' = 0$,

$$y' = Ay + B\alpha = 0. \tag{5}$$

The input beam is collimated, which means $\alpha = 0$. Plugging in the corresponding element from the ABCD matrix (Eq. 3) gives:

$$Ay = 0 \tag{6}$$
$$1 - x = 0 \tag{7}$$
$$x = 1 \tag{8}$$

The solution is x = 1 cm.

The solution can also be found from the Lensmaker equation,

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right]. \tag{9}$$

Plugging in $R_1 = \infty$, $R_2 = 1$, $n = 2$ and $d = 1$ gives the solution f = 1 cm.

Gaussian q-parameter

Recall that the q-parameter is defined as,

$$\frac{1}{q_i} = \frac{1}{R_i} - \frac{i}{\rho_i}, \tag{10}$$

where we have defined the parameter $\rho_i = \pi w_i^2/\lambda$. The initial beam is collimated, $R_i = \infty$, which leads to:

$$q_i = i\rho_i. \tag{11}$$

Now we must propagate our collimated Gaussian beam through our optical system and find the new q-parameter using the ABCD matrix method:

$$\frac{1}{q_0} = \frac{D/q_i + C}{B/q_i + A} = \frac{D + i\rho_i C}{B + i\rho_i A}. \tag{12}$$

We know that at the waist, the real part of $1/q_o$ should be zero since there $R_0 = \infty$. So, multiplying Eq.12 by its complex conjugate and setting the real part to zero leads to the equation,

$$0 = BD + \rho_i^2 AC. \tag{13}$$

Plugging in the the corresponding elements from the system ABCD matrix (Eq. 3) and solving for x gives:

$$0 = \frac{1 + x}{4} + \rho_i^2(x - 1) \tag{14}$$

$$x = \frac{4\rho_i^2 - 1}{4\rho_i^2 + 1} = \frac{4\pi^2 - 1}{4\pi^2 + 1} = 0.95 \text{ [cm]} \tag{15}$$

2 Focusing problem: is the waist dependent of the index of the medium traversed?

Consider the sequence lens-air-interface glass-air, length of glass L, terminated by an interface glass-air at the focus. Find the dependence ofthe focal spot size and intensity on the index of the glass.

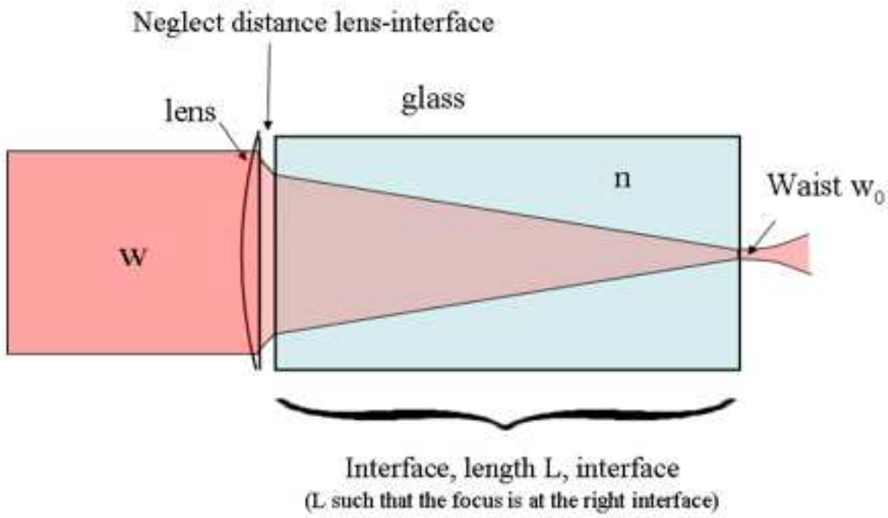

Figure 2:

Solution

Beam Waist dependence

In this situation we are ignoring the length of air that is between the lens and glass plate. The ABCD matrix for this scenario (thin lens, interface, length L, interface) is:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 - \frac{L}{nf} & \frac{L}{n} \\ -\frac{1}{f} & 1 \end{bmatrix}. \tag{16}$$

We start from a collimated beam $q_1 = i\rho_1$, to which we apply the $ABCD$ matrix (Eq. 16) to find q_0 at the focus:

$$\frac{1}{q_0} = \frac{D + i\rho_1 C}{B + i\rho_1 A} = \frac{BD + \rho_1^2 AC + i\rho_1(BC - AD)}{B^2 + \rho_1^2 A^2}. \tag{17}$$

At the waist, the real part of this expression should be zero, leading to the length L to the waist:

$$L = \frac{nf\rho_1^2}{\rho_1^2 + f^2}. \tag{18}$$

In the limit of $\rho_1 \gg f$ we find indeed $L \approx nf$ which we know to be correct. For the imaginary part of Eq. (17) we find:

$$Im\left[\frac{1}{q_0}\right] = \frac{1}{\rho_0} = \frac{\rho_1(BC - AD)}{B^2 + \rho_1^2 A^2} \tag{19}$$

$$= \frac{\rho_1}{\left(\frac{L}{n}\right)^2 + \rho_1^2\left(1 - \frac{L}{nf}\right)^2} \tag{20}$$

Notice that the determinant (AD-BC) equals 1 which must be true since we start and end in the same medium. We now plug in the L that puts us at the waist (Eq. 18), which gives:

$$\rho_0 = \frac{\left(\frac{f\rho_1^2}{\rho_1^2+f^2}\right)^2 + \rho_1^2\left(1 - \frac{\rho_1^2}{\rho_1^2+f^2}\right)^2}{\rho_1} \tag{21}$$

We can again check the limit of $\rho_1 \gg f$ which we find indeed simplifies to $\rho_0 = f^2/\rho_1$. Plugging in for the ρ's gives:

$$\frac{f}{w_1} = \frac{\pi w_0}{\lambda} = \frac{1}{\theta}, \tag{22}$$

which we know to be correct. ***Conclusion:*** the beam waist is independent of n in this case.

Intensity dependence

Because the beam waist size isn't dependent on the index, it may be tempting to conclude that the intensity also does not depend on the index. Intensity is Power divided by Area after all! But, while the area is maintained, the power will not be maintained due to the Fresnel loss at each interface. Because the Fresnel reflection coefficients are indeed highly dependent on n, we must conclude that the Intensity **is** dependent on the index.