1 Huygens Integral

The Huygens integral is constructed on the principle that each point of a wavefront is the source of spherical waves. In fact, in making such a statement, one is using the fact that Maxwell's equation in vacuum and in a linear dielectric medium are identical, with the difference that ϵ_0 is replaced by ϵ. In all textbooks (Born & Wolf, Pedrotti) you will see the classical derivation starting from a point source (a concept predating the laser era, based on the fact that only a tiny pinhole could provide a source with some degree of coherence), which makes a spherical wave reaching an aperture of a particular shape. From this aperture, one proceeds taking each point as a source of spherical waves. It makes more sense to start from an arbitrary wavefront. An infinite uniform plane wave has a field distribution $\mathcal{E}(x_0, y_0, z_0)$ at z_0. Each point of the z_0 plane is considered to be a source of spherical wave issued from P_0. In the following equations $\rho(r-r_0)$ represents the distance between point P_0 and the point P where we want to calculate the field:

$$\rho(r-r_0) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}.$$

The spherical wave issued from P_0 at P is:

$$\mathcal{E} = \frac{\mathcal{E}_0 e^{ik\rho(r-r_0)}}{\rho(r-r_0)}. \tag{1}$$

Huygens's integral is the sum of these spherical waves. See Pedrotti for the justification of the normalization factor $-i/\lambda$.

$$\begin{aligned} \mathcal{E}(x,y,z) &= -\frac{i}{\lambda}\int\int_{S_0} \frac{\mathcal{E}(x_0,y_0,z_0)e^{-ik\rho(r-r_0)}}{\rho(r-r_0)} K(\theta) dS_0, \\ &= -\frac{i}{L\lambda}\int\int_{S_0} \frac{\mathcal{E}(x_0,y_0,z_0)e^{-ik\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}}{L} K(\theta) dS_0 \end{aligned} \tag{2}$$

Refer to Eq. (1)

1.1 Obliquity factor

An "obliquity factor $K[\theta(r, r_0)]$"has been added to the integral. It has to take into account that an infinite plane wave does not diffract, and that the radiation goes only forward, and not backward in the absence of an interface. Note that the units of all field are V/m in Eq. (2). Most textbooks, instead of starting from a field distribution, start with a point source field in V/m^2.

Why an obliquity factor $K(\theta)$?

Wavelets issued from a plane: the light could go forward or backward?

Consider a plane wave incident on a medium with linear polarization:

$$P = \epsilon_0 \chi E \tag{3}$$

A plane wave hits the plane of dipoles located at z_0. The field at z_0 is:

$$E(z_0, t) = \mathcal{E}_0 e^{i(\omega t - kz_0)} \tag{4}$$

The re-emission at $t = 0$ from the dipoles in the forward direction is proportional to:

$$\mathcal{E}_0 e^{-ikz_0} e^{i[\omega t - k(z - z_0)]} = \mathcal{E}_0 e^{i[\omega t - kz]} \tag{5}$$

There is no longer a dependence on z_0. In an uniform dielectric, we have to integrate Eq. (5) over z_0, and the contribution from all planes add up in phase.

$$K(0) = 1$$

In the backward direction:

$$\mathcal{E}_0 e^{-ikz_0} e^{i[\omega t + k(z - z_0)]} = \mathcal{E}_0 e^{-2ik_0} e^{i\omega t - kz} \tag{6}$$

This expression averages to zero in a *uniform* macroscopic medium.

$$K(\pi) = 0$$

This is not the case for instance, if the medium consists of Multiple Quantum Well that can be absorbing, have gain, or simply dielectric. If the QW are dielectrics spaced by one wavelength λ, the $\exp[-2ik_0] = 1$, and we have a case where

$$K(\pi) = 1$$

A function that matches both directions in an *homogeneous, isotropic* medium is the oblicity factor used conventionally:

$$K(\theta) = \frac{1 + \cos\theta}{2}. \tag{7}$$

2 Fresnel approximation

2.1 From the Huygens integral

Fresnel approximation, in one dimension, consists in making in the Huygen's integral the substitution:

$$K(\theta) \approx 1$$
$$\rho(r - r_0) \approx (z - z_0) + \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)}. \quad (8)$$

The approximation results from simply making the approximation: $(x-x_0)/(z-z_0) \ll 1$ and expanding the square root:

$$\rho(r-r_0) = \sqrt{(z - z_0)^2 + (x - x_0)^2 + (y - y_0)^2} \approx (z-z_0)\left[1 + \frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)}\right] \quad (9)$$

The Fresnel integral in one dimension is:

$$\mathcal{E}(x) = \sqrt{\frac{i}{L\lambda}} \int_{-\infty}^{\infty} \mathcal{E}(x_0) e^{-ik(x-x_0)^2/2L} dx_0 \quad (10)$$

Note that this integral is a convolution product of the two functions:

$$\mathcal{E}(x_0)$$

and

$$e^{-ik(x_0)^2/2L} = e^{-i\frac{k}{2L}(x_0)^2}$$

. The Fourier transform of the latter function is

$$e^{-ik_x^2 L/2k}$$

If we apply the convolution theorem, we find, taking the Fourier transform of Eq. (10) (without the normalization factor)

$$\tilde{\mathcal{E}}(k_x, L) = \tilde{\mathcal{E}}(k_x, 0) e^{-ik_x^2 L/2k} \quad (11)$$

which is the paraxial approximation derived directly from Maxwell's equations (see Section 2.2).

2.2 Paraxial approximation

The paraxial wave approximation consist in making the approximation that the beam is propagating mostly along the z direction in Maxwell's wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{\mathcal{E}}(x, y, z)e^{i(\omega t - kz)} = 0. \tag{12}$$

Neglecting the term in ∂^2 we find:

$$2ik\frac{\partial\tilde{\mathcal{E}}(x, y, z)}{\partial z} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{\mathcal{E}}(x, y, z). \tag{13}$$

taking the spatial Fourier transform:

$$2ik\frac{\partial\tilde{\mathcal{E}}(k_x, k_y, z)}{\partial z} = \left(k_x^2 + k_y^2\right)\tilde{\mathcal{E}}(k_x, k_y, z), \tag{14}$$

which can be integrated to yield:

$$\tilde{\mathcal{E}}(k_x, k_y, z) = \tilde{\mathcal{E}}(k_x, k_y, 0)e^{-i\frac{k_x^2 + k_y^2}{2k}z} \tag{15}$$

which is the 2 D extension of the expression (11).

3 Fraunhofer approximation

The exponential factor in the Fresnel integral (10) can be written:

$$\begin{aligned} -ik(x-x_0)^2/2L &= -i\frac{k}{2L}x^2 - i\frac{k}{2L}x_0^2 + ik(\frac{x}{L})x_0 \\ &= \left[-i\frac{k}{2L}x^2\right] - i\frac{k}{2L}x_0^2 + ik_x x_0. \end{aligned} \tag{16}$$

We have changed the Fresnel integral into a Fourier transform:

$$\tilde{\mathcal{E}}(k_x, L) = \sqrt{\frac{i}{L\lambda}} e^{-i\frac{k}{2L}x^2} \int_{-\infty}^{\infty} \left[\mathcal{E}(x_0) e^{-i\frac{k}{2L}x_0^2}\right] e^{ik_x x_0} dx_0. \tag{17}$$

This last equation shows that, in substituting for the field $\tilde{\mathcal{E}}(x_0)$ into the Fraunhofer approximation the expression $[\mathcal{E}(x_0)e^{-i\frac{k}{2L}x_0^2}]$, we obtain the Fresnel approximation. In the Fraunhofer approximation, all exponential factors in x_i^2/L are neglected, leaving:

$$\tilde{\mathcal{E}}(k_x, L) = \sqrt{\frac{i}{L\lambda}} \int_{-\infty}^{\infty} \left[\mathcal{E}(x_0) e^{ik_x x_0}\right] dx_0. \tag{18}$$