Polarization summarized

1 Cartesian representation

1.1 Linear Polarization

Polarization will in general be modified by propagation through an optical system (can be a crystal, liquid, interferometer) that is not isotropic. The axis to be chosen should be related to that optical system.

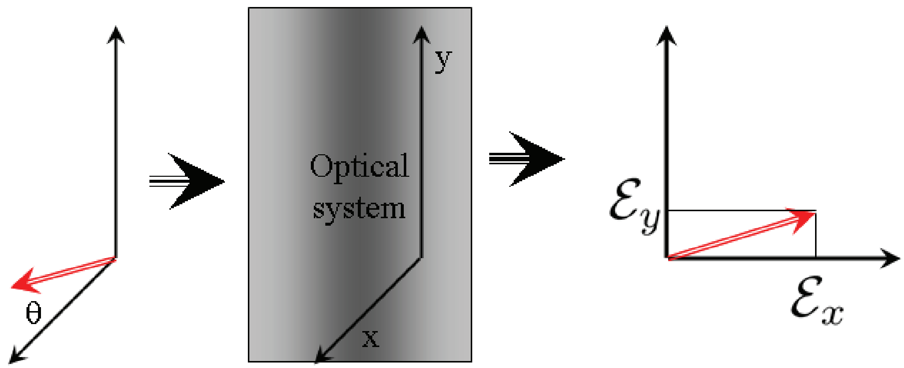

Figure 1: An initial (linear) polarization is sent through an optical system. The optical system has some anisotropy associated with it, that will define the axis that we will chose to represent all polarization.

The initial field in the axis defined by the optical system is defined by the projections of the field on the axis:

$$\begin{aligned} E_x &= \mathcal{E}_x \cos \omega t \\ E_y &= \mathcal{E}_y \cos \omega t \end{aligned}$$

where we choose the phase of the field to be zero on both axis, to define linear polarization.

The ratio of $\mathcal{E}_y/\mathcal{E}_x$ determines the orientation of the electric field vector.

$$\theta \arctan \frac{\mathcal{E}_y}{\mathcal{E}_x}.$$

Dichroism

The polarization direction can be changed by attenuating selectively either the x or y component. Dichroism is the selective absorption of one of two orthogonal components. Control of polarization by selective losses of one component include:

- Wire polarizer (E-field zero in a conductor $\rightarrow$ the grid passes the E-field orthogonal to the wires.
- sheet polarizer (molecular version of the wire polarizer)
- Crystals with color center
- Brewster angle — no reflection in the plane of incidence
- Dielectric mirrors (non normal incidence)

Brewster angle: $\arctan n$. At Brewster angle, the reflected and refracted rays are orthogonal.

Geometric Polarization rotation

A periscope can change the polarization from horizontal to vertical, simply based on the fact that the electric field vector has to remain orthogonal to the k vector.

Different phase shifts along x and y

There are numerous optical components/systems that will produce a difference phase shift along two orthogonal directions. The most common is a crystal, such as quartz. The transmission through crystals with a different index of refraction parallel to the optics axis and perpendicular to the optics axis gives:

$$\begin{aligned} E_x &= \mathcal{E}_x \cos(\omega t - k_x z) \\ E_y &= \mathcal{E}_y \cos(\omega t - k_y z) \end{aligned} \tag{1}$$

where $k_x = 2\pi n_x/\lambda$ and $k_y = 2\pi n_y/\lambda$. The two different indices of refraction result in a difference phase for the E_x and E_y components after propagation through a thickness d of material. A parallel face plate for which the difference in phase is $(k_y - k_x)d = (2N+1)\pi/2$ is called a quarter wave plate. Taking for instance $k_x d = 2N\pi$, then according to Eq. (1) the transmitted field is:

$$\begin{aligned} E_x &= \mathcal{E}_x \cos(\omega t) \\ E_y &= \mathcal{E}_y \sin(\omega t) \end{aligned} \tag{2}$$

This is the equation of the field describing a circle (if $\mathcal{E}_x = \mathcal{E}_y$) at an angular velocity ω. A parallel face plate for which the difference in phase is $(k_y - k_x)d = (2N+1)\pi$ is called a half wave plate. Taking for instance $k_x d = 2N\pi$, then according to Eq. (1) the transmitted field is:

$$\begin{aligned} E_x &= \mathcal{E}_x \cos(\omega t) \\ E_y &= -\mathcal{E}_y \cos(\omega t) \end{aligned} \tag{3}$$

The electric field is linearly polarized, rotated by 90^o. Such a 90^o rotation occurs if the optics axis of the waveplate is at 45^o to the original linear polarization. The half wave plate rotates a linear polarization, by twice the angle between the optics axis of the plate and the original polarization vector. The operation is reversible: two passages through a half wave plate (reflecting the beam back with a mirror) result in the original polarization.

A quarter wave plate transforms the linear to circular if the optic axis of the waveplate is at 45^o to the original polarization. Otherwise, the linear polarization is changed to elliptical. Two quarter wave plates amount to a half wave. Thus by reflecting the beam into the waveplate, one should have a rotation of the polarization by 90^o is the optic axis is at 45^o.

N is the order of the waveplate. In general, N is a large number. A zero order waveplate is difficult to manufacture. Waveplates are usually only good for a narrow wavelength range.

An electro-optical modulator is a waveplate induced by an electric field. An electric field can also orient molecules (Kerr cell). The Kerr cell was used to demonstrate the first Q-switched ruby laser. The orientation can be much faster using intense and ultrashort light pulses. A fast shutter can be made on this principle.

Stresses also introduce anisotropy in indices. This is used in fiber (nonlinear polarization mode-locking).

Phase shifts on reflection

Instead of the phase shifts $\phi_x = k_x d$ and $\phi_y = k_y d$, ϕ_x and ϕ_y can simply be the phase shifts in reflection. In that case, x may be the plane of incidence, and y normal to that plane (or vice versa). A large differential phase shift takes place in total internal reflection. A corner cube glass reflector generally transforms linear polarized light into near circular polarized light. A Rhomb prism used two total internal reflections at 45^o from the plane of incidence to transform linear polarization into circular polarization. These prisms have the advantage over quarter wave plates of being broadband.

The Fresnel Rhomb

In many circumstances, we need to control the polarization of beams accurately (say for example to obtain optimal contrast in an interferometer). There are a number of optical elements that can be used for this purpose. Their common feature is to introduce a phase change between two orthogonally polarized light beams. This can be done, for example, by utilizing the optical birefringence in crystals (which leads to the waveplates) *or* using total internal reflection (TIR) at the *interface between glass and air*. Do not confuse TIR with "critical angle". TIR occurs at any angle above the critical angle. The reflection amplitude is then always unity (= 100%), but the phase shift is a steep function of the angle of incidence (above the critical angle).

The incident light is linearly polarized with the axis tilted 45° with respect to the edges of the rhomb's input face (i.e. half $\perp$, half $||$). The light undergoes two total reflections before it leaves the rhomb each giving a $\pi/4$ phase shift. *Again, total reflection does not mean that you have to be at the critical angle.* Let us calculate the refractive index n which the glass has to have for the output light to be circularly polarized. Assume a symmetrical beam path as shown in Fig. 2 (with $\beta = 90°$) and $\alpha = 45°$.

The angle of incidence (in the glass of index n) is θ. On the other side of the interface, the index is 1, and the angle is θ_2 which is given by Snell's law:

$$n_2 \sin\theta_2 = \sin\theta_2 = n \sin\theta \tag{4}$$

and

$$\cos\theta_2 = \sqrt{1 - n^2 \sin^2\theta} = i\sqrt{n^2 \sin^2\theta - 1} \tag{5}$$

This implies that you have to replace $\sin\theta_2$ by $n\sin\theta$ and $\cos\theta_2$ by $i\sqrt{n^2\sin^2\theta - 1}$ in Fresnel equations, which become then complex numbers. The two Fresnel equations have then the form:

$$r_\parallel = \frac{\cos(\theta) - n\cos\theta_2}{\cos(\theta) + n\cos\theta_2} = \frac{a - ib}{a + ib} = \frac{y}{y^*}, \tag{6}$$

$$r_\perp = \frac{n\cos(\theta) - \cos\theta_2}{n\cos(\theta) + \cos\theta_2} = \frac{c - id}{c + id} = \frac{z}{z^*} \tag{7}$$

It is easy to verify that $|r|^2 = 1$ for both parallel and perpendicular condition, independently of the angle provided $\theta \geq \theta_{\text{critical}}$

One notes that Eqs. (6) and (7) is the ratio of a complex number and its complex conjugate, and thus the phase $\phi = 2\delta$ where δ is the phase shift of y or z. The simple approach is to compute $\tan(\delta_\parallel - \delta_\perp) = \tan \pm\pi/4 = \pm 0.41421$ We have:

$$\tan\delta_\parallel = \frac{-n\sqrt{n^2\sin^2\theta - 1}}{\cos\theta}$$

$$\tan\delta_\perp = \frac{-\sqrt{n^2\sin^2\theta - 1}}{n\cos\theta}$$

The tangent of the difference angle is:

$$\tan(\delta_\parallel - \delta_\perp) = \frac{\tan\delta_\parallel - \tan\delta_\perp}{1 + \tan\delta_\perp \tan\delta_\parallel}$$

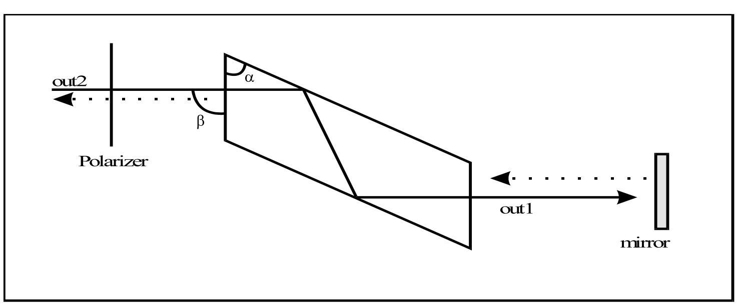

Figure 2: The Fresnel Rhomb

$$
\begin{aligned}
&= \frac{(-n^2+1)\sqrt{n^2\sin^2\theta-1}n\cos\theta\left[1+\frac{n(n^2\sin^2\theta-1)}{n\cos^2\theta}\right]}{} \\
&= \frac{-(1-n^2)\cos\theta\sqrt{n^2\sin^2\theta-1}}{n(n^2-1)\sin^2\theta} \\
&= \frac{-\cos\theta\sqrt{n^2\sin^2\theta-1}}{n\sin^2\theta} \\
&= -\frac{-\sqrt{n^2-2}}{n}
\end{aligned} \tag{8}
$$

where the last equation (8) is for the value $\theta = \pi/4$. Taking the square of Eq. (8) yields a simple second order equation in n, which has as solution

$$
n = \frac{t^2+\sqrt{t^4=8}}{2} = 1.5025, \tag{9}
$$

where $t = \tan\pi/4 = 0.4142$.

2 General ellipse: ellipticity and orientation

We take the general polarization ellipse to be given by:

$$\begin{aligned} E_x &= \frac{1}{2}\tilde{\mathcal{E}}_1 e^{i\omega t} + c.c. = \mathcal{E}_1 \cos(\omega t + \varphi_1) \\ E_y &= \frac{1}{2}\tilde{\mathcal{E}}_2 e^{i\omega t} + c.c. = \mathcal{E}_2 \cos(\omega t + \varphi_2) \end{aligned} \tag{10}$$

The equation of the ellipse is obtained by eliminating $\sin\omega t$ and $\cos\omega t$ from the 2 Eqs. (10).

$$\begin{aligned} \frac{E_x}{\mathcal{E}_1} &= \cos\omega t\cos\varphi_1 - \sin\omega t\sin\varphi_1 \\ \frac{E_y}{\mathcal{E}_2} &= \cos\omega t\cos\varphi_2 - \sin\omega t\sin\varphi_2 \\ \frac{E_x}{\mathcal{E}_1}\sin\varphi_2 - \frac{E_y}{\mathcal{E}_2}\sin\varphi_1 &= \cos\omega t\sin\delta \\ \frac{E_x}{\mathcal{E}_1}\cos\varphi_2 - \frac{E_y}{\mathcal{E}_2}\cos\varphi_1 &= \sin\omega t\sin\delta, \end{aligned} \tag{11}$$

where $\delta = \varphi_2 - \varphi_1$. Taking the sum of the squares eliminates the terms in ωt, yielding:

$$|\frac{E_x}{\mathcal{E}_1}|^2 + |\frac{E_y}{\mathcal{E}_2}|^2 - 2\frac{E_x E_y}{\mathcal{E}_1\mathcal{E}_2}\cos\delta = \sin^2\delta. \tag{12}$$

That is the equation of an ellipse not aligned with the axis.

The ellipse referred to its principal axis (ξ, η) has the equation

$$\begin{aligned} E_\xi &= a\cos(\omega t + \varphi_0) \\ E_\eta &= b\sin(\omega t + \varphi_0) \end{aligned} \tag{13}$$

The transformation of axis is the rotation:

$$\begin{aligned} E_\xi &= E_x\cos\psi + E_y\sin\psi \\ E_\eta &= -E_x\sin\psi + E_y\cos\psi \end{aligned} \tag{14}$$

In these axis we have indeed:

$$|\frac{E_\xi}{a}|^2 + |\frac{E_\eta}{b}|^2 = 1. \tag{15}$$

We plug the ellipse Eqs. (15) into Eqs. (14):

$$\begin{aligned} E_\xi &= a(\cos(\omega t)\cos\varphi_0 - \sin\omega t\sin\varphi_0) \\ &= \mathcal{E}_1\left(\cos\omega t\cos\varphi_1 - \sin\omega t\sin\varphi_1\right)\cos\psi + \mathcal{E}_2\left(\cos\omega t\cos\varphi_2 - \sin\omega t\sin\varphi_2\right)\sin\psi \\ E_\eta &= b(\sin\omega t\cos\varphi_0 + \cos\omega t\sin\varphi_0) \\ &= -\mathcal{E}_1\left(\cos\omega t\cos\varphi_1 - \sin\omega t\sin\varphi_1\right)\sin\psi + \mathcal{E}_2\left(\cos\omega t\cos\varphi_2 - \sin\omega t\sin\varphi_2\right)\cos\psi \end{aligned} \tag{16}$$

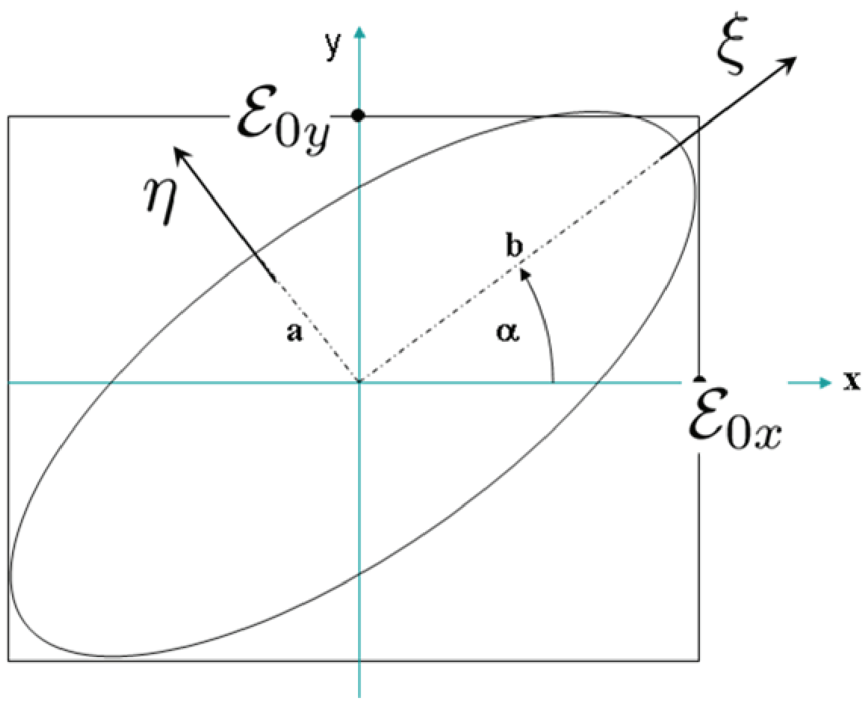

Figure 3: Polarization ellipse in lab frame (x, y), and in the frame of the ellipse (ξ, η).

Equate the coefficients of $\sin \omega t$ and $\cos \omega t \to 4$ equations. Manipulations of these 4 equations will be the clue to the final result. For the first two equations:

$$a \cos \varphi_0 = \mathcal{E}_1 \cos \varphi_1 \cos \psi + \mathcal{E}_2 \cos \varphi_2 \sin \psi \tag{17}$$

$$a \sin \varphi_0 = \mathcal{E}_1 \sin \varphi_1 \cos \psi + \mathcal{E}_2 \sin \varphi_2 \sin \psi \tag{18}$$

Taking the sum of the squares:

$$a^2 = \mathcal{E}_1^2 \cos^2 \psi + \mathcal{E}_2^2 \sin^2 \psi + 2\mathcal{E}_1 \mathcal{E}_2 \sin \psi \cos \psi \cos \delta \tag{19}$$

For the next two equations (16):

$$b \cos \varphi_0 = \mathcal{E}_1 \sin \varphi_1 \sin \psi - \mathcal{E}_2 \sin \varphi_2 \cos \psi \tag{20}$$

$$b \sin \varphi_0 = -\mathcal{E}_1 \cos \varphi_1 \sin \psi + \mathcal{E}_2 \cos \varphi_2 \cos \psi \tag{21}$$

Taking the sum of the squares:

$$b^2 = \mathcal{E}_1^2 \sin^2 \psi + \mathcal{E}_2^2 \cos^2 \psi - 2\mathcal{E}_1 \mathcal{E}_2 \sin \psi \cos \psi \cos \delta \tag{22}$$

The sum of Eqs (19) and (22) gives:

$$a^2 + b^2 = \mathcal{E}_1^2 + \mathcal{E}_2^2. \tag{23}$$

Another manipulation of the equations is to multiply Eq. (17) by (20) and Eq. (18) by (21) to construct $ab \cos^2 \varphi_0 + ab \sin^2 \varphi_0$ to find:

$$ab = -\mathcal{E}_1 \mathcal{E}_2 \sin \delta \tag{24}$$

An additional equation is obtained by finding from the equations two expression for the ratio b/a [(20)/(17) and (21)/(18)]:

$$\begin{aligned}\frac{b}{a} &= \frac{\mathcal{E}_1 \sin\varphi_1 \sin\psi - \mathcal{E}_2 \sin\varphi_2 \cos\psi}{\mathcal{E}_1 \cos\varphi_1 \cos\psi + \mathcal{E}_2 \cos\varphi_2 \sin psi} \\ &= \frac{-\mathcal{E}_1 \cos\varphi_1 \sin\psi + \mathcal{E}_2 \cos\varphi_2 \cos\psi}{\mathcal{E}_1 \sin\varphi_1 \cos\psi + \mathcal{E}_2 \sin\varphi_2} \end{aligned} \tag{25}$$

which, after multiplying by the denominators, gives:

$$(\mathcal{E}_1^2 - \mathcal{E}_2^2)\sin 2\psi = 2\mathcal{E}_1\mathcal{E}_2 \cos 2\psi, \tag{26}$$

which gives us the inclination angle of the ellipse:

$$\boxed{\tan 2\psi = \frac{2\mathcal{E}_1\mathcal{E}_2}{(\mathcal{E}_1^2 - \mathcal{E}_2^2)}.} \tag{27}$$

From Eqs. (23) and (24) we get:

$$\boxed{\begin{aligned} a &= \frac{1}{2}\left[\sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_1\mathcal{E}_2 \sin\delta} + \sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2 - \mathcal{E}_1\mathcal{E}_2 \sin\delta}\right] \\ b &= \frac{1}{2}\left[\sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_1\mathcal{E}_2 \sin\delta} - \sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2 - \mathcal{E}_1\mathcal{E}_2 \sin\delta}\right] \qquad (28) \end{aligned}}$$

3 Circular basis

Let us assume a real axis along x and the imaginary axis along y.

$$E = \mathcal{E}\left\{a_R e^{i\omega t + i\pi/4} + a_L e^{-i\omega t + i\pi/4}\right\} \tag{29}$$

This is equivalent to the cartesian form. Let us assume equal circular polarizations, i.e. $a_R = a_L$. The x and y components are then:

$$\begin{aligned} E_x &= \mathrm{Re}\{E\} = \mathcal{E}\cos\frac{\pi}{4}\cos\omega t \\ E_y &= \mathrm{Imag}\{E\} = \mathcal{E}\sin\frac{\pi}{4}\cos\omega t \end{aligned}$$

which is indeed equivalent to the cartesian representation of the linear polarization.

The representation for waves propagating through a transparent medium:

$$E = \mathcal{E}\left\{a_R e^{i\omega t + i\pi/4 - k_R z} + a_L e^{-i\omega t + i\pi/4 - k_L z}\right\} \tag{30}$$

In general:

$$\boxed{E = a_R \tilde{e}_R + a_L \tilde{e}_L} \tag{31}$$

Relation to the cartesian coordinates:

$$\boxed{\begin{aligned} a_R &= \frac{1}{2}\sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2 - \mathcal{E}_1\mathcal{E}_2\sin\delta} \\ a_L &= \frac{1}{2}\sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_1\mathcal{E}_2\sin\delta} \end{aligned} \qquad (32)}$$

To understand this relation, let us look at the example of Fig. 4. The black ellipse can be decomposed into an x component of amplitude 2, and a y-component of amplitude 1 lagging by $\pi/2$. It can also be decomposed into a counterclockwise rotating vector of length 1.5 (red) and a clockwise rotating vector of length 0.5 (blue). The right circular amplitude is thus $(a+b)/2$ and the left circular $(a-b)/2$, where a and b are the major and minor axis of the ellipse. This can also easily be seen from Eq. (13) of the ellipse in its axis:

$$\begin{aligned} E_\xi + iE_\eta &= \frac{a}{2}\left(e^{i(\omega t+\varphi_0)} + e^{-i(\omega t+\varphi_0)}\right) + i\frac{b}{2i}\left(e^{i(\omega t+\varphi_0)} - e^{-i(\omega t+\varphi_0)}\right) \\ &= \frac{a+b}{2}e^{i(\omega t+\varphi_0)} + \frac{a-b}{2}e^{i(\omega t+\varphi_0)} \\ &= a_R e^{i(\omega t+\varphi_0)} + a_L e^{-i(\omega t+\varphi_0)} \end{aligned} \tag{33}$$

Equation (32) is the general case where we have substituted the expressions (28) for a and b.

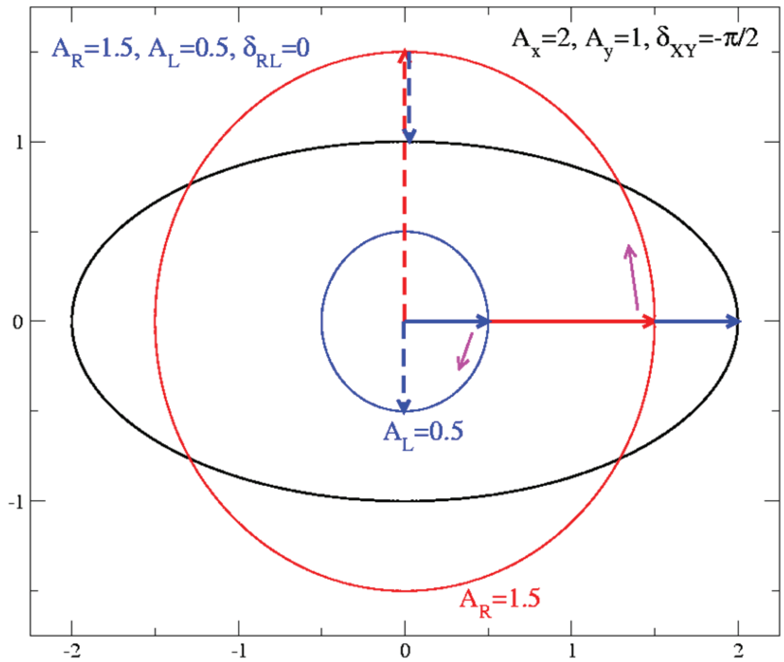

Figure 4: Black ellipse = polarization ellipse. It is decomposed in x and y components, or counter- and clockwise components

Rotation of linear polarization Linearly polarized light making an angle θ with the axis x would be represented by:

$$\begin{aligned} E_\xi &= \cos\theta\cos\omega t \\ E_\eta &= \sin\theta\cos\omega t \\ E_\xi + iE_\eta &= e^{i\theta}\cos\omega t = \frac{1}{2}\left[e^{i(\omega t+\theta)} + e^{-i(\omega t-\theta)}\right] \end{aligned} \tag{34}$$

which indicates opposite phase shift of the right and left circular polarization.

All the effects listed in the section on cartesian representation have their equivalent here. Some materials have different absorption coefficient for left and right circular polarized beams. The equivalent of the Dichroism that changed k_x versus k_y, is "optical activity", or a different k_R and k_L. This will result in a rotation of the plane of polarization for linearly polarized light.

The most common device based on a difference in k_R and k_L is the Faraday rotator. A magnetic field is applied along the axis of propagation k of the light. There is a sense of rotation of free charges around the magnetic field ($ev \times B$). A different index is seen by the field rotating in the same direction as the charges, as compared to the counter-rotating field. The plane of polarization is rotated by this device, as in the case of the waveplate. There is however an essential difference. If light is retro-reflected back through the Faraday rotator, the polarization continues its rotation with respect to the magnetic field vector (not with respect to the k vector). In the case of the waveplate, the retro-reflected light regains its original polarization. It is therefore possible with a Faraday rotator to construct an optical isolator. If the Faraday rotator rotates the polarization 45^o, the retro-reflected light will be rotated 90^o, and can be eliminated with a polarizing beam splitter.

3.1 Change in circular polarization by geometry

The polarization is changed from right circular to left circular upon retro-reflection, because the k-vector is reversed, and the polarization is defined with respect to the k-vector. The polarization is however unchanged if the light is reflected by two metallic mirrors forming a $90^{\rm o}$ angle.

4 Stokes parameters (1852)

We take the general polarization ellipse to be given by:

$$\begin{aligned}
E_x &= \frac{1}{2}\tilde{\mathcal{E}}_1 e^{i\omega t} + c.c. = \mathcal{E}_1 \cos(\omega t + \varphi_1) \\
E_y &= \frac{1}{2}\tilde{\mathcal{E}}_2 e^{i\omega t} + c.c. = \mathcal{E}_2 \cos(\omega t + \varphi_2)
\end{aligned} \tag{35}$$

Only the phase difference $\delta - \varphi_2 - \varphi_1$ is relevant. The Stokes parameters are defined by:

$$\begin{aligned}
I = s_0^2 &= \mathcal{E}_1^2 + \mathcal{E}_2^2 \\
Q = s_1^2 &= \mathcal{E}_1^2 - \mathcal{E}_2^2 \\
U = s_2^2 &= 2\mathcal{E}_1\mathcal{E}_2 \cos\delta \\
V = s_3^2 &= 2\mathcal{E}_1\mathcal{E}_2 \sin\delta
\end{aligned} \tag{36}$$

Only 3 are independent since $s_0^2 = s_1^2 + s_2^2 + s_3^2$. In terms of circular coordinates:

$$\begin{aligned}
I = s_0^2 &= a_R^2 + a_L^2 \\
Q = s_1^2 &= 2a_R a_L \cos\delta \\
U = s_2^2 &= 2a_R a_L \sin\delta \\
V = s_3^2 &= a_R^2 - a_L^2
\end{aligned} \tag{37}$$

The terms I, Q, U, V refer to spherical coordinates Q, U, V on a sphere of radius I:

$$\begin{aligned}
Q &= I \cos 2\chi \cos 2\psi \\
U &= I \cos 2\chi \sin 2\psi \\
V &= I \sin 2\chi
\end{aligned} \tag{38}$$

ψ is the inclination of the ellipse as defined previously. χ is the ellipticity defined as $\chi = \arctan b/a$. This is the Poincare sphere, to represent a polarization state (really equivalent to the Stokes parameters). Useful in crystal optics.

The Stokes parameters are also manipulated by matrices (the Mueller matrices). All these representations have some shortcoming. Stokes parameter are time and space averaged. and are intensity measurements. Jones matrices (1941) ignore the intensity, but are local and adequate to treat interferences.

Maybe it is best to stick to the local — in time and space — field description?

4.1 Examples

Linear polarization $\mathbf{V = 0}$. Why? $\delta = o$, or $a_R = a_L$.
$U = 0$ along the axis
$Q = 0$ at 45 deg.

Poincare sphere: in the plane $U - -Q$.

Circular polarization $V = \pm I$, rest is zero.
Poincare sphere: the poles (right circular = north pole, left circular = south pole).

5 How to measure the polarization state?

5.1 Direct measurement

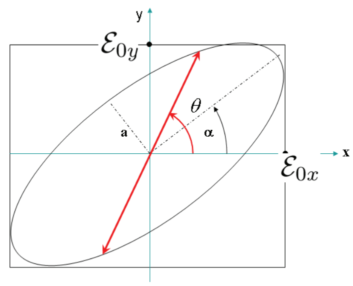

Figure 5: Measuring the polarization ellipse: a linear polarizer is rotated by an angle θ varying over 360 degrees.

The most direct measurement of the polarization ellipse is to scan a linear polarizer around a point. The result of this measurement is a scan of the intensity along the variable direction θ. Taking the square root of that measurement is the polarization ellipse. The normalized polarization ellipse leads to the Jones Matrix.

5.2 Measurement of Stokes parameters

See Schaefer et al, American Journal of Physics **75**, 163 (2007).

6 Modulation of polarization: Polarization gating

A multiple order quarter plate wave combined with a polarizer can be used as a frequency filter, just as a Fabry-Perot. Another application of a multiple order plate is to modulate the polarization of an ultrashort optical pulse. As will be shown in the following example, a quarter wave plate designed for the center wavelength and the middle of the pulse maybe a half wave plate for the leading and trailing edges of the pulse.

The chosen example is that of a crystalline quartz plate, through which a 35 fs duration pulse at 800 nm is sent. At that wavelength, the extraordinary and ordinary indices of quartz are $n_e = 1.53838$, and $n_o = 1.54727$. On calculates that the corresponding wave vectors are $k_e = 12.5\mu\text{m}^{-1}$ and $k_o = 12.08\mu\text{m}^{-1}$. If we choose a thickness of quartz of 1065.75 μm, the difference in optical path for the two axis is $\Delta nd = 11.75\lambda$. This implies a high order quarter wave plate. One can also consider the phase difference for the two axis:

$$\Delta kd = 2\pi\frac{\Delta nd}{\lambda} = 23.5 \times \pi = 47\frac{\pi}{2}$$

which implies a 47th order quarter wave plate.

Along the two axis, there is also a difference in transit time of $\Delta nd/c = 31.3$ fs.

The 35 fs pulse has a bandwidth of $0.0938 \cdot 10^{15}$ s^{-1}. Let us consider the two points of the pulse spectrum at half width from the center frequency, i.e. at $\pm\Delta\Omega = 0.0469 \cdot 1015$ s^{-1}, and calculate the difference in retardation with respect to the central frequency. The dispersion of the extraordinary and ordinary k vector is:
$\frac{dk_e}{d\Omega} = 5.21 \cdot 10^{-15}$ s μm^{-1}
$\frac{dk_0}{d\Omega} = 5.18 \cdot 10^{-15}$ s μm^{-1}
with a difference $\Delta k' = 0.03 \cdot 10^{-15}$ s μm^{-1}. The difference in retardation with respect to the central frequency is thus:

$$\pm\Delta\Omega \times \Delta k' \times d = \Delta\varphi = 0.03 \times 0.0469 \times 1065.75 = 1.5 \approx \frac{pi}{2}$$

. We have thus a $46 \times \pi/2$ (half wave) ahead of the pulse, which becomes a $46 \times \pi/2$ (quarter wave) in the middle, and a $48 \times \pi/2$ (half wave) at the end.

Following with a zero order quarter wave, that gives a pulse that changes its polarization from circular to linear to circular. This has been used in the generation of attosecond pulses. After tunnel ionization, re-collision can only occur for linearly polarized light.

All the above are rather coarse approximation, but it give on an idea of the principle of polarization gating.

7 Nonlinear waveplates

The index of refraction is intensity dependent. This is called the Kerr optical effect, in analogy with the Kerr electro-optic effect, in which the change of index in a liquid (CS_2 for instance) is proportional to an electric field, as a result of molecular alignment. The Kerr effect has in general an electronic component (practically instantaneous) and a molecular component (molecular orientation). We write the index of refraction as: $n = n_0 + n_2|E|^2$ or $n = n_0 + \bar{n}_2 I$. In terms of polarization:

$$P^{NL}(\omega) = \epsilon_0 \chi^{(3)} |E|^2 E = \epsilon_0 \chi^{(3)} E(\omega) E(-\omega) E(\omega). \tag{39}$$

Typical value for n_2 is 10^{-16} cm^2/W. For air, a bit less: $5 \cdot 10^{-19}$ cm^2/W. Note that this latter value is not that insignificant: you can have short pulses propagating as self-induced waveguides (filaments) with an intensity of $2 \cdot 10^{13}$ W/cm^2. The phase shift over a distance of ℓ is $\Delta n = 2\pi \Delta n \ell/\lambda = 10{-}4 \times 10^4 \times \ell(\text{cm})$ which is huge! Air as been demonstrated as a "waveplate" by sending a strong pump pulse and a probe pulse polarized at 45o.

The third order susceptibility is not a scalar but a tensor:

$$P_i = \epsilon_0 \chi^{(3)}_{ijk\ell} E_j E_k E_\ell \tag{40}$$

We consider degenerate interactions of the type $\chi(\omega, \omega, -\omega, \omega)$ and an isotropic medium, which means that the coefficients $\chi^{(3)}_{ijk\ell}$ have to be independent of axis permutations. The result of the "isotropy" condition, the nonlinear polarization can be written [1]

$$P = \epsilon_0 6\chi_{1122}(E \cdot E^*)E + 3\epsilon_0 \chi_{1221}(E \cdot E)E^* = \epsilon_0 A(E \cdot E^*)E + \frac{1}{2}\epsilon_0 B(E \cdot E)E^*. \tag{41}$$

Coefficients "A" and "B" are "historical" notations [2], and are related to nonlinear susceptibilities through $A = 6\chi_{1122}$ and $B = 6\chi_{1221}$. For molecular alignment $B = 6A$; for electronic Kerr effect $B = A$, and $B = 0$ for electrostriction effect.

Since we are interested in propagation of elliptically polarized beam, it is more convenient to present the field in the basis of of circular polarization.

$$E = E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_- \tag{42}$$

where

$$\begin{aligned} \hat{\sigma}_+ &= \frac{\hat{x} + i\hat{y}}{\sqrt{2}} \\ \hat{\sigma}_- &= \frac{\hat{x} - i\hat{y}}{\sqrt{2}} \end{aligned} \tag{43}$$

It is easy to verify the following set of equations:

$$\begin{aligned} \hat{\sigma}^*_\pm &= \hat{\sigma}_\mp \\ \hat{\sigma}^{\cdot}_\pm \hat{\sigma}_\pm &= 0 \\ \hat{\sigma}^{\cdot}_\pm \hat{\sigma}_\mp &= 1. \end{aligned} \tag{44}$$

We can use these relations in the products appearing in Eq. (41).

$$\begin{aligned} E^* \cdot E &= (E^*_+ \hat{\sigma}^*_+ + E^*_- \hat{\sigma}^*_-) \cdot (E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) = E^*_+ E_+ + E^*_- E_- = |E_+|^2 + |E_-|^2 \\ E \cdot E &= (E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) \cdot (E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) = E_+ E_- + E_- E_+ = 2E_+ E_- \end{aligned} \tag{45}$$

Substituting in Eq. (41), there will be a right circular component of the polarization and a left circular one:

$$P_{NL}\epsilon_0 A\left(|E_+|^2+|E_-|^2\right)(E_+\hat{\sigma}_+ + E_-\hat{\sigma}_-) + 2\epsilon_0 B(E_+E_-)(E_+^*\hat{\sigma}_- + E_+^*\hat{\sigma}_-)$$

The polarization can also be written in terms of right and left circular components: $P_N L = P_+\hat{\sigma}_+ + P_-\hat{\sigma}_-$:

$$P_\pm = \epsilon_0 A|E_\pm|^2 E_\pm + \epsilon_0(A+B)|E_\mp|^2 E_\pm^*, \tag{46}$$

Using the wave equation for plane wave propagation, the nonlinear polarization leads to the following change in index:

$$n_\pm \sim n_0 + \frac{1}{n_0}[A|E_\pm|^2 + (A+B)|E_\mp|^2]. \tag{47}$$

The first consequence of Eq. (eq:deltan) is that the axis of the ellipse will rotate:

$$n_+ - n_- = \frac{B}{n_0}\left(|E_-|^2 - |E_+|^2\right). \tag{48}$$

The rotation is largest when the Kerr effect is due to molecular orientation (larger B coefficient - in fact B = 6A in that case). If the polarization is linear, $|E_-|^2 = |E_+|^2$ and there can be no rotation. If the polarization is circular.. obviously the "long axis" a circle does not rotate. There is only an E_+ or an E_-, hence only $n_\pm = (A/n_0)|E_+|^2$

In the case of elliptically polarized light, the birefringence results in a change of field propagation for the two components of E_+ and E_-. The two senses of rotation will therefore focus at different rates. The weaker circular component increases in intensity, making the circular polarization become more linear. One can thus conclude that circular polarization is unstable.

Angular momentum

The elementary particle corresponding to circularly polarized light is the photon, a spinning field at frequency ω. We have seen that to the photon is associated an energy $\hbar\omega$, and a linear momentum $\hbar\omega/c$. What about the angular momentum? It value is $\hbar$. This corresponds to a linear momentum of $\hbar\omega/c \times$ and arm of $\lambda/(2\pi)$. The angular momentum associated with a circularly polarized pulse of energy W is:

$$N\hbar = \frac{W}{\hbar\omega} \times \hbar = \frac{W}{\omega} \tag{49}$$

As in the case of the linear momentum, the angular momentum of the beam does not depend on $\hbar$, The kinetic energy associated with the beam of power P is P/ω.

One can construct beams with an helicoidal wavefront, such that the rays generate a hyperboloid. Each ray will have a velocity component in a plane orthogonal to the propagation direction of the beam. Each photon has thus associated with it a momentum component in a plane normal to the propagation equal to:

$$\frac{\hbar\omega}{c} \sin\theta$$

where θ is the inclination of the ray with respect to the propagation axis of the beam. If that ray is at a distance r from the axis, there is an "orbital" angular momentum associated with each photon:

$$\frac{\hbar\omega}{c} \sin\theta \times r.$$

This momentum is larger than the intrinsic angular momentum per photon ($\hbar$), since the "arm" r can be of the order of mm, rather than micron ($\lambda/2\pi$). The kinetic energy in rotation associated with such a beam of power P is

$$\frac{P}{\hbar\omega}\frac{\hbar\omega}{c} \sin\theta r = \frac{P}{c} \sin\theta r.$$

References

[1] W. Boyd. *Nonlinear Optics*. Academic Press, New York, 1991.

[2] P. D. Maker and R. W. Terhune. Study of optical effects due to an induced polarization third order in the electric field strength. *Phys. Rev.*, 137(3A):A801–A818, Feb 1965.