1 Analogy between pulse and beam propagation

1.1 Time analogy of the paraxial approximation

Comparing the paraxial wave equation and the reduced wave equation describing pulse propagation through a GVD medium we notice an interesting correspondence. Both equations are of similar structure. In terms of the reduced wave equation the transverse space coordinates x, y seem to play the role of the time variable. This space-time analogy suggests the possibility of translating simply the effects related to dispersion into beam propagation properties. For instance, we may compare the temporal broadening of an unchirped pulse due to dispersion with the change of beam size due to diffraction. In this sense free-space propagation plays a similar role for the beam characteristics as a GVD medium does for the pulse envelope. The field spectrum at a distance z is:

$$\tilde{\mathcal{E}}(k_x, z) = \tilde{\mathcal{E}}(k_x, z = 0)e^{i(k_x^2+k_y^2)z/(2k_\ell)}. \tag{1}$$

In time, the spectral envelope after propagation through a thickness z of a linear transparent material is given by:

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{\mathcal{E}}(\Omega, 0)e^{-\frac{i}{2}k_\ell''\Omega^2 z} \tag{2}$$

The exponential phase factor $(k_x^2+k_y^2)z/(2k)$ which describes transverse beam diffraction in space, corresponds to the exponential phase factor $-k''\Omega^2 z/2$ which describes pulse dispersion in time. There is a correspondence $-k'' \to 1/k$. A difference that brings some complication is the dimensionality: 2 dimensions in space versus one dimension in time.

Since Eq. (1) corresponded to the paraxial approximation, the analogy can be carried over to successive subsets of that approximation. It will thus apply also to Gaussian optics, and the time equivalent of the Fraunhofer and geometric approximations.

1.2 Gaussian beams and Gaussian pulses

An important particular solution of the wave equation within the paraxial approximation is the Gaussian beam (see, e.g., [1]). In order to understand better the space-time analogy for Gaussian beams/pulses, the derivation is reproduced below.

1.2.1 Gaussian beams

We look for a solution to the wave equation:

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} + \frac{i}{2k}\left(\frac{\partial^2 \tilde{\mathcal{E}}}{\partial x^2} + \frac{\partial^2 \tilde{\mathcal{E}}}{\partial y^2}\right) = 0 \tag{3}$$

of the form:

$$\tilde{\mathcal{E}} = \mathcal{E}_0 e^{-i\left[P+\frac{k}{2q}(x^2+y^2)\right]} \tag{4}$$

Substituting (only the coefficients of the common exponentials are written below):

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} = -i\frac{dP}{dz} + i\frac{kr^2}{2q^2}\frac{dq}{dz}, \tag{5}$$

where we have written r^2 for $x^2 + y^2$.

$$\frac{\partial \tilde{\mathcal{E}}}{\partial x} = -i\frac{kx}{q}. \tag{6}$$

Taking the second derivative:

$$\frac{\partial^2 \tilde{\mathcal{E}}}{\partial x^2} + \frac{\partial^2 \tilde{\mathcal{E}}}{\partial y^2} = -2i\frac{k}{q} - \frac{k^2}{q^2}r^2. \tag{7}$$

The factor 2 in the first term of the right hand side is the main difference arise between the two dimensional space and the one dimensional time situations. Substituting in the wave Eq. (3), and equating the terms of same power in r:

$$i\frac{dP}{dz} = \frac{1}{q} \tag{8}$$

$$\frac{dq}{dz} = 1 \tag{9}$$

The last equation gives us:

$$q = q_0 + z \tag{10}$$

We separate $1/q$ in a real and imaginary part:

$$\frac{1}{q} = \frac{1}{R} - \frac{i}{\rho} \tag{11}$$

We choose the z axis such that at $z = 0$ $q_0 = i\rho_0$ is purely imaginary ($R = \infty$). Separating real and imaginary parts in Eq. (10):

$$R = R(z) = z + \rho_0^2/z \tag{12}$$

$$\rho = \rho_0(1 + z^2/\rho_0^2) \tag{13}$$

$$\tag{14}$$

Equation (8) can be integrated, leading to

$$P = -i\ln\left(\frac{q_0 + z}{q_0}\right)$$

and

$$e^{-iP} = \frac{q_0}{q_0 + z} = \frac{1}{1 - i\frac{z}{\rho_0}} = \frac{1}{\sqrt{1 + \frac{z^2}{\rho_0^2}}}e^{-\Phi(z)} = \frac{w}{w_0}e^{-\Phi(z)}, \tag{15}$$

where Φ is the Guoy phase shift,

$$\Theta = \Theta(z) = \arctan(z/\rho_0). \tag{16}$$

Up to Eq. (15), everything can be transposed to the time domain.

1.2.2 Gaussian Pulses

We look for a solution to the wave equation:

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} - \frac{ik''}{2}\left(\frac{\partial^2 \tilde{\mathcal{E}}}{\partial t^2}\right) = 0 \tag{17}$$

of the form:

$$\tilde{\mathcal{E}} = \mathcal{E}_0 e^{-i\left[Q + \frac{1}{2p}(t^2)\right]} \tag{18}$$

Substituting (only the coefficients of the common exponentials are written below):

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} = -i\frac{dQ}{dz} + i\frac{t^2}{2p^2}\frac{dp}{dz}, \tag{19}$$

where we have now t^2 instead of $x^2 + y^2$.

$$\frac{\partial \tilde{\mathcal{E}}}{\partial t} = i\frac{t}{p}. \tag{20}$$

Taking the second derivative:

$$\frac{\partial^2 \tilde{\mathcal{E}}}{\partial t^2} = \frac{i}{k''p} - \frac{t^2}{p^2}. \tag{21}$$

The factor 2 in the first term of the right hand side is no longer there because of the difference in dimensionality. Substituting in the wave Eq. (17), and equating the terms of same power in t:

$$i\frac{dQ}{dz} = \frac{1}{2p} \tag{22}$$

$$\frac{dp}{dz} = 1 \tag{23}$$

The last equation gives us:

$$p = p_0 + z \tag{24}$$

We separate $1/p$ in a real and imaginary part:

$$\frac{1}{p} = \frac{1}{R} - \frac{i}{\sigma} \tag{25}$$

We choose the z axis such that at $z = 0$, $q_0 = i\sigma_0$ is purely imaginary ($R = \infty$). Separating real and imaginary parts in Eq. (24):

$$R = R(z) = z + \sigma_0^2/z \tag{26}$$

$$\sigma = \sigma_0(1 + z^2/\sigma_0^2) \tag{27}$$

$$\tag{28}$$

Equation (22) can be integrated, leading to

$$2Q = -i\ln\left(\frac{p_0 + z}{p_0}\right)$$

and

$$e^{-iQ} = \sqrt{\frac{p_0}{p_0 + z}} = \frac{1}{\sqrt{1 - i\frac{z}{\sigma_0}}} = \frac{1}{[1 + \frac{z^2}{4\sigma_0^2}]^{1/4}}e^{-\Phi(z)/2} = \frac{\tau_{G0}}{\tau_G}e^{-\Phi(z)/2}, \tag{29}$$

where Φ is the Guoy phase shift,

$$\Theta = \Theta(z) = \arctan(z/2\sigma_0). \tag{30}$$

Now we have to define R and σ in terms of the pulse duration and phase. If we choose:

$$\sigma = \frac{\tau_G^2}{2}, \tag{31}$$

the propagation Eq. (24) is satisfied. For the focal distance associated with a phase modulation:

$$R = \frac{1}{\ddot{\varphi}}. \tag{32}$$

Replacing Eq. (31) and (32) in Eq. (18) we find indeed the correct temporal dependence of the Gaussian amplitude and phase.

1.3 Review of ABCD matrix in space

An ABCD matrix [1] is a ray transfer matrix which describes the effect of an optical element on a laser beam. It can be used both in geometrical optics and for propagating Gaussian beams. The paraxial approximation is always required for ABCD matrix calculations. Tracing of a light path through an optical system can then be performed by multiplying an element matrix by a vector representing the light ray:

$$\begin{pmatrix} y_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix} \tag{33}$$

where y and α refer to transverse displacement and offset angle from an optical axis respectively. The subscripts '1' and '2' denote the coordinates before and after an optical element. For example, a thin lens with focal length f has the following ABCD matrix:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \tag{34}$$

and propagation through free space over a distance d is associated with the matrix:

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{35}$$

Transition from space to time Geometric optics can be seen as an approximation of Gaussian propagation, where the propagation distance is much larger than the Rayleigh range. Therefore, the propagation in a linear medium is in a straight line making an angle α with the optics axis:

$$w = w_0\sqrt{1+\left(\frac{z}{\rho_0}\right)^2} \approx \frac{w_0}{\rho_0}z = \alpha z = \frac{2}{k_\ell w_0}z. \tag{36}$$

The same approximation can be made in the time domain to define the "optical inclination" α_t:

$$\tau = \tau_0\sqrt{1+\left(\frac{z}{L_d}\right)^2} \approx \frac{\tau_0}{L_d}z = \alpha_t z = \frac{2k''_\ell}{\tau_{G0}}z. \tag{37}$$

Note that in contrast to the space domain where α is dimensionless, α_t has the dimension of the inverse of a velocity. The generic operation of Eq. (33) has its time domain equivalent:

$$\begin{pmatrix} T_2 \\ \alpha_{t,2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ \alpha_{t,1} \end{pmatrix} \tag{38}$$

where T, the correspondent of the transverse displacement y, is a temporal position. In the time equivalent of the propagation matrix (35), the distance d is replaced by $k''_\ell d$, while the time equivalent of the lens matrix (34) has the element $-1/f$ replaced by an imposed chirp $\ddot{\Phi}$ which, for instance, in the case of Kerr modulation, is equal to:

$$\ddot{\Phi} = \frac{8\pi\ell_{Kerr}}{\lambda}n_2\frac{I_0}{\tau_G^2}, \tag{39}$$

where ℓ_{Kerr} is the length of the nonlinear medium characterized by an intensity dependent index $n_2 I$.

1.4 ABCD matrix in time

Propagation (which is dispersion instead of diffraction):

$$\begin{pmatrix} 1 & k''d \\ 0 & 1 \end{pmatrix} \tag{40}$$

Lensing (which is am imposed quadratic phase modulation with a "curvature" $\ddot{\Phi}$)

$$\begin{pmatrix} 1 & 0 \\ -\ddot{\Phi} & 1 \end{pmatrix}, \tag{41}$$

Kerr phase modulation

$$\begin{pmatrix} 1 & 0 \\ -\frac{8\pi\ell_{Kerr}}{\lambda} n_2 \frac{I_0}{\tau_G^2} & 1 \end{pmatrix}, \tag{42}$$

The various level of approximation of space-time analogy are summarized in Table 1.

1.5 Gaussian pulses as analogue of Gaussian beams

As we have seen in the previous section the quadratic phase factor in Eq. (2) broadens an unchirped input pulse and leads to a (linear) frequency sweep across the pulse (chirp) while the pulse spectral width (and shape) remains unchanged. A "bandwidth-limited" Gaussian beam means a beam without phase variation across the beam, which requires a radius of curvature of the phase front $R = \infty$. Thus, a Gaussian beam is "bandwidth-limited" at its waist where it takes on its minimum possible size (at a given spatial frequency spectrum). Multiplication with a quadratic phase factor to describe the beam propagation, leads to beam broadening and "chirp." The latter simply accounts for a finite phase front curvature. Roughly speaking, the spatial frequency components which are no longer needed to form the broadened beam profile are responsible for the beam divergence.

Space		**Time**
$\left(\frac{\partial^2}{\partial x^2} - 2ik_\ell \frac{\partial}{\partial z}\right)\mathcal{E}(x,z) = 0$		$\left(\frac{\partial}{\partial z} + \frac{n}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{n}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(\Omega,z) = 0$
$\left(\frac{\partial}{\partial z} - \frac{i(k_x^2+k_y^2)}{2k_\ell}\right)\tilde{\mathcal{E}}(x,z) = 0$	Fourier transform	$\left(\frac{\partial}{\partial z} + ik_\ell(\Omega)\right)\mathcal{E}(\Omega,z) = 0$
$\mathcal{E}(k_x,z) = \mathcal{E}(k_x,0)\,\exp\left(\frac{i}{2k_\ell}(k_x^2+k_y^2)z\right)$		$\mathcal{E}(\Omega,z) = \mathcal{E}(\Omega,0)\,\exp\left(-\frac{ik_\ell''}{2}\Omega^2 z\right)$
$\frac{k_x^2+k_y^2}{2k_\ell}$	$\Longleftrightarrow$	$-\frac{k_\ell''\Omega^2}{2}$
$\tilde{\mathcal{E}}(x,z) \propto$ $\int_{-\infty}^{\infty}\tilde{\mathcal{E}}(x_0,0)\,\exp\left(i\frac{k_\ell x}{L}x_0\right)dx_0$	Fraunhofer approximation	$\tilde{\mathcal{E}}(t,z) \propto$ $\int_{-\infty}^{\infty}\tilde{\mathcal{E}}(t_0,0)\,\exp\left(i\frac{t}{k_\ell'' z}t_0\right)dt_0$
$\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}$	Displacement matrix	$\begin{pmatrix}1 & k_\ell''\,d\\ 0 & 1\end{pmatrix}$
$\begin{pmatrix}1 & 0\\ -\frac{1}{f} & 1\end{pmatrix}$	Lens matrix	$\begin{pmatrix}1 & 0\\ -\ddot{\phi} & 1\end{pmatrix}$

Table 1: Space-time equivalence, starting from the Fourier transform of Maxwell's equation in space (left) and in time (right).

References

[1] H. W. Kogelnik and T. Li. Laser beams and resonators. *Appl. Opt.*, 5:1550–1567, 1966.