1 Analogy between pulse and beam propagation

1.1 Time analogy of the paraxial approximation

Comparing the paraxial wave equation and the reduced wave equation describing pulse propagation through a GVD medium we notice an interesting correspondence. Both equations are of similar structure. In terms of the reduced wave equation the transverse space coordinates x, y seem to play the role of the time variable. This space-time analogy suggests the possibility of translating simply the effects related to dispersion into beam propagation properties. For instance, we may compare the temporal broadening of an unchirped pulse due to dispersion with the change of beam size due to diffraction. In this sense free-space propagation plays a similar role for the beam characteristics as a GVD medium does for the pulse envelope. The field spectrum at a distance z is:

$$\tilde{\mathcal{E}}(k_x, z) = \tilde{\mathcal{E}}(k_x, z = 0)e^{i(k_x^2+k_y^2)z/(2k_\ell)}. \tag{1}$$

In time, the spectral envelope after propagation through a thickness z of a linear transparent material is given by:

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{\mathcal{E}}(\Omega, 0)e^{-\frac{i}{2}k_\ell''\Omega^2 z} \tag{2}$$

The exponential phase factor $(k_x^2+k_y^2)z/(2k)$ which describes transverse beam diffraction in space, corresponds to the exponential phase factor $-k''\Omega^2 z/2$ which describes pulse dispersion in time. There is a correspondence $-k'' \to 1/k$. A difference that brings some complication is the dimensionality: 2 dimensions in space versus one dimension in time.

Since Eq. (1) corresponded to the paraxial approximation, the analogy can be carried over to successive subsets of that approximation. It will thus apply also to Gaussian optics, and the time equivalent of the Fraunhofer and geometric approximations.

1.2 Gaussian beams and Gaussian pulses

An important particular solution of the wave equation within the paraxial approximation is the Gaussian beam (see, e.g., [1]). In order to understand better the space-time analogy for Gaussian beams/pulses, the derivation is reproduced below.

1.2.1 Gaussian beams

We look for a solution to the wave equation:

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} + \frac{i}{2k}\left(\frac{\partial^2 \tilde{\mathcal{E}}}{\partial x^2} + \frac{\partial^2 \tilde{\mathcal{E}}}{\partial y^2}\right) = 0 \tag{3}$$

of the form:

$$\tilde{\mathcal{E}} = \mathcal{E}_0 e^{-i\left[P + \frac{k}{2q}(x^2+y^2)\right]} \tag{4}$$

Substituting (only the coefficients of the common exponentials are written below):

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} = -i\frac{dP}{dz} + i\frac{kr^2}{2q^2}\frac{dq}{dz}, \tag{5}$$

where we have written r^2 for $x^2 + y^2$.

$$\frac{\partial \tilde{\mathcal{E}}}{\partial x} = -i\frac{kx}{q}. \tag{6}$$

Taking the second derivative:

$$\frac{\partial^2 \tilde{\mathcal{E}}}{\partial x^2} + \frac{\partial^2 \tilde{\mathcal{E}}}{\partial y^2} = -2i\frac{k}{q} - \frac{k^2}{q^2} r^2. \tag{7}$$

The factor 2 in the first term of the right hand side is the main difference arise between the two dimensional space and the one dimensional time situations. Substituting in the wave Eq. (3), and equating the terms of same power in r:

$$i\frac{dP}{dz} = \frac{1}{q} \tag{8}$$

$$\frac{dq}{dz} = 1 \tag{9}$$

The last equation gives us:

$$q = q_0 + z \tag{10}$$

We separate $1/q$ in a real and imaginary part:

$$\frac{1}{q} = \frac{1}{R} - \frac{i}{\rho} \tag{11}$$

We choose the z axis such that at $z = 0$ $q_0 = i\rho_0$ is purely imaginary ($R = \infty$). Separating real and imaginary parts in Eq. (10):

$$R = R(z) = z + \rho_0^2/z \tag{12}$$

$$\rho = \rho_0(1 + z^2/\rho_0^2) \tag{13}$$

$$\tag{14}$$

Equation (8) can be integrated, leading to

$$P = -i \ln\left(\frac{q_0 + z}{q_0}\right)$$

and

$$e^{-iP} = \frac{q_0}{q_0 + z} = \frac{1}{1 - i\frac{z}{\rho_0}} = \frac{1}{\sqrt{1 + \frac{z^2}{\rho_0^2}}} e^{-\Phi(z)}, \tag{15}$$

where Φ is the Guoy phase shift,

$$\Theta = \Theta(z) = \arctan(z/\rho_0). \tag{16}$$

Up to Eq. (15), everything can be transposed to the time domain, with the parameters having the *same dimension*.

1.2.2 Gaussian Pulses

We look for a solution to the wave equation:

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} - \frac{ik''}{2}\left(\frac{\partial^2 \tilde{\mathcal{E}}}{\partial t^2}\right) = 0 \tag{17}$$

of the form:

$$\tilde{\mathcal{E}} = \mathcal{E}_0 e^{-i\left[Q+\frac{1}{2k''p}(t^2)\right]} \tag{18}$$

Substituting (only the coefficients of the common exponentials are written below):

$$\frac{\partial \tilde{\mathcal{E}}}{\partial z} = -i\frac{dQ}{dz} + i\frac{t^2}{2k''p^2}\frac{dp}{dz}, \tag{19}$$

where we have now t^2 instead of $x^2 + y^2$.

$$\frac{\partial \tilde{\mathcal{E}}}{\partial t} = i\frac{t}{k''p}. \tag{20}$$

Taking the second derivative:

$$\frac{\partial^2 \tilde{\mathcal{E}}}{\partial t^2} = \frac{i}{k''p} - \frac{t^2}{k'' * 2p^2}. \tag{21}$$

The factor 2 in the first term of the right hand side is no longer there because of the difference in dimensionality. Substituting in the wave Eq. (17), and equating the terms of same power in t:

$$i\frac{dQ}{dz} = \frac{1}{2p} \tag{22}$$

$$\frac{dp}{dz} = 1 \tag{23}$$

The last equation gives us:

$$p = p_0 + z \tag{24}$$

We separate $1/p$ in a real and imaginary part:

$$\frac{1}{p} = \frac{1}{R} - \frac{i}{\sigma} \tag{25}$$

We choose the z axis such that at $z = 0$ $q_0 = i\sigma_0$ is purely imaginary ($R = \infty$). Separating real and imaginary parts in Eq. (24):

$$R = R(z) = z + \sigma_0^2/z \tag{26}$$

$$\sigma = \sigma_0(1 + z^2/\sigma_0^2) \tag{27}$$

$$\tag{28}$$

Equation (22) can be integrated, leading to

$$2Q = -i\ln\left(\frac{2p_0 + z}{2p_0}\right)$$

and

$$e^{-iQ} = \sqrt{\frac{2p_0}{2p_0 + z}} = \frac{1}{\sqrt{1 - i\frac{z}{2\sigma_0}}} = \frac{1}{\left(1 + \frac{z^2}{4\sigma_0^2}\right)^{1/4}} e^{-\Theta(z)/2} = \frac{\tau_{G0}}{\tau_G} e^{-\Theta(z)/2}, \tag{29}$$

where Φ is the Guoy phase shift,

$$\Theta = \Theta(z) = \arctan(z/2\sigma_0). \tag{30}$$

Now we have to define R and σ in terms of the pulse duration and phase. If we choose:

$$\sigma = \frac{\tau_G^2}{2k''}, \tag{31}$$

the propagation Eq. (24) is satisfied. For the focal distance associated with a phase modulation:

$$R = \frac{1}{\ddot{\varphi} k''}. \tag{32}$$

Replacing Eq. (31) and (32) in Eq. (18) we find indeed the correct temporal dependence of the Gaussian amplitude and phase.

References

[1] H. W. Kogelnik and T. Li. Laser beams and resonators. *Appl. Opt.*, 5:1550–1567, 1966.