In the case of a linear medium, the forward and backward wave travel independently. If, as initial condition, we choose $\tilde{\mathcal{E}}_B = 0$ along the line $r + s = 0$ ($t = 0$), there will be no back scattered wave. If the polarization is written as a slowly varying amplitude:

$$\tilde{P} = \frac{1}{2}\tilde{\mathcal{P}}_F e^{i\omega_\ell s} + \frac{1}{2}\tilde{\mathcal{P}}_B e^{i\omega_\ell r}, \tag{1.110}$$

the equations for the forward and backward propagating wave also separate if $\tilde{\mathcal{P}}_F$ is only a function of $\tilde{\mathcal{E}}_F$, and $\tilde{\mathcal{P}}_B$ only a function of $\tilde{\mathcal{E}}_B$. This is because a source term for $\tilde{\mathcal{P}}_B$ can only be formed by a "grating" term, which involves a product of $\tilde{\mathcal{E}}_B\tilde{\mathcal{E}}_F$. It applies to a polarization created by near resonant interaction with a two-level system, using the semi-classical approximation, as will be considered in Chapters 3 and **??**. The separation between forward and backward travelling waves has been demonstrated by Eilbeck [17, 18] outside of the slowly-varying approximation. Within the slowly varying approximation, we generally write that the second derivative with respect to time of the polarization as $-\omega_\ell^2\tilde{\mathcal{P}}$, and therefore, the forward and backward propagating waves are still uncoupled, even when $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(\tilde{\mathcal{E}}_F, \tilde{\mathcal{E}}_B)$, provided there is only a forward propagating beam as initial condition.

1.2.3 Dispersion

For nonzero GVD ($k_\ell'' \neq 0$) the propagation problem (1.96) can be solved either directly in the time or in the frequency domain. In the first case, the solution is given by a Poisson-integral [19] which here reads

$$\tilde{\mathcal{E}}(t,z) = \frac{1}{\sqrt{2\pi i k_\ell'' z}} \int_{-\infty}^{t} \tilde{\mathcal{E}}(t', z=0) \exp\left(i\frac{(t-t')^2}{2k_\ell'' z} \right) dt' \tag{1.111}$$

As we will see in subsequent chapters, it is generally more convenient to treat linear pulse propagation through transparent linear media in the frequency domain, since only the phase factor of the envelope $\tilde{\mathcal{E}}(\Omega)$ is affected by propagation.

It follows directly from the solution of Maxwell's equations in the frequency domain [for instance Eqs. (1.77) and (1.82)] that the spectral envelope after propagation through a thickness z of a linear transparent material is given by:

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{\mathcal{E}}(\Omega, 0) \exp\left(-\frac{i}{2}k_\ell''\Omega^2 z - \frac{i}{3!}k_\ell'''\Omega^3 z - \ldots \right). \tag{1.112}$$

Thus we have for the temporal envelope

$$\tilde{\mathcal{E}}(t, z) = \mathcal{F}^{-1}\left\{ \tilde{\mathcal{E}}(\Omega, 0) \exp\left(-\frac{i}{2}k_\ell''\Omega^2 z - \frac{i}{3!}k_\ell'''\Omega^3 z - \ldots \right) \right\}. \tag{1.113}$$

If we limit the Taylor expansion of k to the GVD term k''_ℓ, we find that an initially bandwidth-limited pulse develops a spectral phase with a quadratic frequency dependence, resulting in chirp.

We had defined a "chirp coefficient"

$$\kappa_c = 1 + \frac{M^4}{4\langle t^2\rangle_0^2}\left[\left.\frac{d\phi}{d\Omega}\right|_{\omega_\ell}\right]^2$$

when considering in Section 1.1.5 the influence of quadratic chirp on the uncertainty relation Eq. (1.67) based on the successive moments of the field distribution. In the present case, we can identify the phase modulation:

$$\left.\frac{d\phi}{d\Omega}\right|_{\omega_\ell} = -k''_\ell z \tag{1.114}$$

Since the spectrum (in amplitude) of the pulse $|\tilde{\mathcal{E}}(\Omega, z)|^2$ remains constant [as shown for instance in Eq. (1.112)], the spectral components responsible for chirp must appear at the expense of the envelope shape, which has to become broader.

At this point we want to introduce some useful relations for the characterization of the dispersion. The dependence of a dispersive parameter can be given as a function of either the frequency Ω or the vacuum wavelength λ. The first, second and third order derivatives are related to each other by

$$\frac{d}{d\Omega} = -\frac{\lambda^2}{2\pi c}\frac{d}{d\lambda} \tag{1.115}$$

$$\frac{d^2}{d\Omega^2} = \frac{\lambda^2}{(2\pi c)^2}\left(\lambda^2\frac{d^2}{d\lambda^2} + 2\lambda\frac{d}{d\lambda}\right) \tag{1.116}$$

$$\frac{d^3}{d\Omega^3} = -\frac{\lambda^3}{(2\pi c)^3}\left(\lambda^3\frac{d^3}{d\lambda^3} + 6\lambda^2\frac{d^2}{d\lambda^2} + 6\lambda\frac{d}{d\lambda}\right) \tag{1.117}$$

The dispersion of the material is described by either the frequency dependence $n(\Omega)$ or the wavelength dependence $n(\lambda)$ of the index of refraction. The derivatives of the propagation constant used most often in pulse propagation problems, expressed in terms of the index n, are:

$$\boxed{\begin{aligned}
\frac{dk}{d\Omega} &= \frac{n}{c} + \frac{\Omega}{c}\frac{dn}{d\Omega} = \frac{1}{c}\left(n - \lambda\frac{dn}{d\lambda}\right) && (1.118)\\
\frac{d^2k}{d\Omega^2} &= \frac{2}{c}\frac{dn}{d\Omega} + \frac{\Omega}{c}\frac{d^2n}{d\Omega^2} = \left(\frac{\lambda}{2\pi c}\right)\frac{1}{c}\left(\lambda^2\frac{d^2n}{d\lambda^2}\right) && (1.119)\\
\frac{d^3k}{d\Omega^3} &= \frac{3}{c}\frac{d^2n}{d\Omega^2} + \frac{\Omega}{c}\frac{d^3n}{d\Omega^3} = -\left(\frac{\lambda}{2\pi c}\right)^2\frac{1}{c}\left(3\lambda^2\frac{d^2n}{d\lambda^2} + \lambda^3\frac{d^3n}{d\lambda^3}\right) && (1.120)
\end{aligned}}$$

The second equation, Eq. (1.119), defining the group velocity dispersion (GVD) is the frequency derivative of $1/v_g$. Multiplied by the propagation length L, it describes the frequency dependence of the group delay. It is sometimes expressed in fs^2 μm^{-1}.

A positive GVD corresponds to

$$\frac{d^2k}{d\Omega^2} > 0 \qquad (1.121)$$

1.2.4 Gaussian pulse propagation

For a more quantitative picture of the influence that GVD has on the pulse propagation we consider the linearly chirped Gaussian pulse of Eq. (1.33)

$$\tilde{\mathcal{E}}(t, z = 0) = \mathcal{E}_0 e^{-(1+ia)(t/\tau_{G0})^2} = \mathcal{E}_0 e^{-(t/\tau_{G0})^2} e^{i\varphi(t,z=0)}$$

entering the sample. To find the pulse at an arbitrary position z, we multiply the field spectrum, Eq. (1.35), with the propagator $\exp\left(-i\frac{1}{2}k_\ell''\Omega^2 z\right)$ as done in Eq. (1.112), to obtain

$$\tilde{\mathcal{E}}(\Omega, z) = \tilde{A}_0 e^{-x\Omega^2} e^{iy\Omega^2} \qquad (1.122)$$

where

$$x = \frac{\tau_{G0}^2}{4(1+a^2)} \qquad (1.123)$$

and

$$y(z) = \frac{a\tau_{G0}^2}{4(1+a^2)} - \frac{k_\ell'' z}{2}. \qquad (1.124)$$

$\tilde{A}_0$ is a complex amplitude factor which we will not consider in what follows and τ_{G0} describes the pulse duration at the sample input. The time dependent electric field that we obtain by Fourier transforming Eq. (1.122) can be written as

$$\tilde{\mathcal{E}}(t, z) = \tilde{A}_1 \exp\left\{-\left(1 + i\frac{y(z)}{x}\right)\left(\frac{t}{\sqrt{\frac{4}{x}[x^2 + y^2(z)]}}\right)^2\right\}. \qquad (1.125)$$

Obviously, this describes again a linearly chirped Gaussian pulse. For the "pulse duration" (note $\tau_p = \sqrt{2\ln 2}\,\tau_G$) and phase at position z we find

$$\tau_G(z) = \sqrt{\frac{4}{x}[x^2 + y^2(z)]} \qquad (1.126)$$

and

$$\varphi(t,z) = -\frac{y(z)}{4[x^2+y^2(z)]}t^2. \tag{1.127}$$

Let us consider first an initially unchirped input pulse ($a = 0$). The pulse duration and chirp parameter develop as:

$$\boxed{\begin{aligned} \tau_G(z) &= \tau_{G0}\sqrt{1+\left(\frac{z}{L_d}\right)^2} && (1.128) \\ \frac{\partial^2}{\partial t^2}\varphi(t,z) &= \left(\frac{1}{\tau_{G0}^2}\right)\frac{2z/L_d}{1+(z/L_d)^2}. && (1.129) \end{aligned}}$$

We have defined a characteristic length:

$$\boxed{L_d = \frac{\tau_{G0}^2}{2k_\ell''}.} \tag{1.130}$$

For later reference let also us introduce a so-called dispersive length defined as

$$L_D = \frac{\tau_{p0}^2}{k_\ell''} \tag{1.131}$$

where for Gaussian pulses $L_D \approx 2.77 L_d$. Bandwidth limited Gaussian pulses double their length after propagation of about $0.6 L_D$. For propagation lengths $z \gg L_d$ the pulse broadening of an unchirped input pulse as described by Eq. (1.128) can be simplified to

$$\frac{\tau_G(z)}{\tau_{G0}} \approx \frac{z}{|L_d|} = \frac{2|k_\ell''|}{\tau_{G0}^2}z. \tag{1.132}$$

It is interesting to compare the result of Eq. (1.128) with that of Eq. (1.65), where we used the second moment as a measure for the pulse duration. Since the Gaussian is the shape for minimum uncertainty [Eq. (1.57)], and since $d^2\phi/d\Omega^2 = -k''z$, one can derive the evolution equation for the mean square deviation of a Gaussian pulse in a dielectric medium:

$$\langle t^2\rangle = \langle t^2\rangle_0 + \left.\frac{d^2\phi}{d\Omega^2}\right|_0 \langle\Omega^2\rangle_0 = \langle t^2\rangle = \langle t^2\rangle_0 + \frac{(k'')^2 z^2}{\langle t^2\rangle_0}. \tag{1.133}$$

The latter equations reduces to Eq. (1.128) by substituting the relations between mean square deviations and Gaussian widths [Eq. (1.58)]. If the input pulse is chirped ($a \neq 0$) two different behaviors can occur depending on the relative sign of

a and k''_ℓ. In the case of opposite sign, $y^2(z)$ increases monotonously resulting in pulse broadening, cf. Eq. (1.126). If a and k''_ℓ have equal sign $y^2(z)$ decreases until it becomes zero after a propagation distance

$$z_c = \frac{\tau_{G0}^2 a}{2|k''_\ell|(1+a^2)}. \tag{1.134}$$

At this position the pulse reaches its shortest duration

$$\tau_G(z_c) = \tau_{Gmin} = \frac{\tau_{G0}}{\sqrt{1+a^2}} \tag{1.135}$$

and the time dependent phase according to Eq. (1.127) vanishes. From here on the propagation behavior is that of an unchirped input pulse of duration τ_{Gmin}, that is, the pulse broadens and develops a time-dependent phase. The larger the input chirp ($|a|$), the shorter the minimum pulse duration that can be obtained [see Eq. (1.135)]. The underlying reason is that the excess bandwidth of a chirped pulse is converted into a narrowing of the envelope by chirp compensation, until the Fourier limit is reached. The whole procedure including the impression of chirp on a pulse will be treated in Chapter **??** in more detail.

There is a complete analogy between the propagation (diffraction) effects of a spatially Gaussian beam and the temporal evolution of a Gaussian pulse in a dispersive medium. For instance, the pulse duration and the slope of the chirp follow the same evolution with distance as the waist and curvature of a Gaussian beam, as detailed at the end of this chapter. A linearly chirped Gaussian pulse in a dispersive medium is completely characterized by the position and (minimum) duration of the unchirped pulse, just as a spatially Gaussian beam is uniquely defined by the position and size of its waist. To illustrate this point, let us consider a linearly chirped pulse whose "duration" τ_G and chirp parameter a are known at a certain position z_1. The position z_c of the minimum duration (unchirped pulse) is found again by setting $y = 0$ in Eq. (1.124):

$$z_c = z_1 + \frac{\tau_G^2}{2k''_\ell}\frac{a}{1+a^2} = z_1 + a\frac{\tau_{Gmin}^2}{2k''_\ell}. \tag{1.136}$$

The position z_c is after z_1 if a and k''_ℓ have the same sign[2]; before z_1 if they have opposite sign. All the temporal characteristics of the pulse are most conveniently defined in terms of the distance $L = z - z_c$ to the point of zero chirp, and the minimum duration τ_{Gmin}. This is similar to Gaussian beam propagation where the

[2]For instance, an initially downchirped ($a > 0$) pulse at $z = z_c$ will be compressed in a medium with positive dispersion ($k'' > 0$).

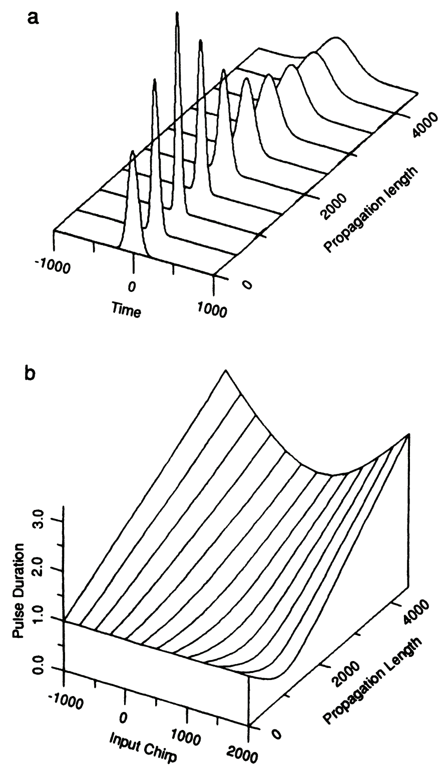

Figure 1.8: Propagation of a linearly chirped Gaussian pulse in a medium with GVD [pulse shape (a), pulse duration for different input chirp (b)].

location of the beam waist often serves as reference. The chirp parameter a and the pulse "duration" τ_G at any point L are then simply given by

$$a(L) = L/L_d \tag{1.137}$$

$$\tau_G(L) = \tau_{Gmin}\sqrt{1+[a(L)]^2} \tag{1.138}$$

where the dispersion parameter $L_d = \tau_{Gmin}^2/(2|k_\ell''|)$. The pulse duration bandwidth product varies with distance L as

$$c_B(L) = \frac{2\ln 2}{\pi}\sqrt{1+[a(L)]^2} \tag{1.139}$$

To summarize, Fig. (1.8) illustrates the behavior of a linearly chirped Gaussian pulse as it propagates through a dispersive sample.

Simple physical consideration can lead directly to a crude approximation for the maximum broadening that a bandwidth limited pulse of duration τ_p and spectral width $\Delta\omega_p$ will experience. Each group of waves centered around a frequency Ω travels with its own group velocity $v_g(\Omega)$. The difference of group velocities over the pulse spectrum becomes then:

$$\Delta v_g = \left[\frac{dv_g}{d\Omega}\right]_{\omega_\ell} \Delta\omega_p. \tag{1.140}$$

Accordingly, after a travel distance L the pulse spread can be as large as

$$\Delta\tau_p = \left|\Delta\left(\frac{L}{v_g}\right)\right| \approx \frac{L}{V_g^2}|\Delta v_g| \tag{1.141}$$

which, by means of Eqs. (1.93) and (1.140), yields:

$$\Delta\tau_p = L|k_\ell''|\Delta\omega_p. \tag{1.142}$$

Approximating $\tau_p \approx \Delta\omega_p^{-1}$, a characteristic length after which a pulse has approximately doubled its duration can now be estimated as:

$$L_D' = \frac{1}{|k_\ell''|\Delta\omega_p^2}. \tag{1.143}$$

Measuring the length in meter and the spectral width in nm the GVD of materials is sometimes given in fs/(m nm) which pictorially describes the pulse broadening per unit travel distance and unit spectral width. From Eq. (1.142) we find for the corresponding quantity

$$\boxed{\frac{\Delta\tau_p}{L\Delta\lambda} = 2\pi\frac{c}{\lambda_\ell^2}|k_\ell''|.} \tag{1.144}$$

For BK7 glass at 620 nm, $k_\ell'' \approx 6.52692 \times 10^{-26}\text{s}^2/\text{m}$, and the GVD as introduced above is about 320 fs per nm spectral width and meter propagation length.

1.2.5 Complex dielectric constant

In general, the dielectric constant, which was introduced in Eq. (1.75) as a real quantity, is complex. Indeed a closer inspection of Eq. (1.74) shows that the finite memory time of matter requires not only ϵ, χ to be frequency dependent but also that they be complex. The real and imaginary part of $\tilde{\epsilon}$, $\tilde{\chi}$ are not independent of each other but related through a Kramers–Kronig relation. The consideration of a real $\epsilon(\Omega)$ is justified as long as we can neglect (linear) losses or gain. This is valid for transparent samples or propagation lengths which are too short for these processes to become essential for the pulse shaping. For completeness we