# Advanced Optics Test 2, Fabry-Perot

Tuesday, December 4, 2018

## 1 Fourier transform

Find the Fourier transform of a triangle of base 2 d (between x = -d to x = d) and height 1.

### 1.1 Solution

The triangle is the convolution of two identical square signals of height 1 and width 2d. The Fourier transform of a convolution is the product of the Fourier transforms of the two function to be convoluted. Thus the solution is a $\mathrm{sinc}^2$.

## 2 Phase on reflection/transmission

You may remember the general rule linking the phase shift on reflection $\varphi_r$, and the phase shift on transmission $\varphi_t$:

$$\varphi_r - \varphi_t = \frac{\pi}{2} \tag{1}$$

which was derived from simple energy conservation. The question is to verify here this rule for a near-resonant Fabry-Perot, given the reflection and transmission functions. To make a simple demonstration, we choose to be slightly off resonance such that the round-trip phase shift in the Fabry-Perot is $\delta = 2N\pi + 0.1$, so that a first order approximation can be made for $\exp(i\delta)$. The reflection coefficient of each mirror is $R = 0.91$.

Verify Eq. (1). You are free to make appropriate approximations. You can also prove it analytically. The equations of the field transfer functions are:

$$\mathcal{T} = \frac{(1-R)e^{i\delta/2}}{1-Re^{i\delta}} \tag{2}$$

for the transmission, and

$$\mathcal{R} = \frac{\sqrt{R}(e^{i\delta}-1)}{1-Re^{i\delta}} \tag{3}$$

for the reflection.

## 2.1 Solution

### 2.1.1 Transmission

The Fabry-Perot transmission function is:

$$\frac{(1-R)e^{i\delta/2}}{1-Re^{i\delta}} \approx \frac{1-R+i\frac{1-R}{2}\delta}{1-R-iR\delta} = \frac{a}{b}e^{i(\varphi_a-\varphi_b)} \tag{4}$$

For the conditions given, $\tan\varphi_a \approx \varphi_a = \delta/2 = 0.05$, and $\tan\varphi_b = R\delta/(1-R) \approx 1$, hence $\varphi_b = \pi/4$. The phase shift on transmission of the Fabry-Perot is thus $\psi_t = 0.05 - \pi/4$.

### 2.1.2 Reflection

The reflection is:

$$\frac{\sqrt{R}(e^{i\delta}-1)}{1-Re^{i\delta}} \approx \frac{\sqrt{R}(i\delta+\delta^2/2}{1-R-iR\delta} = \frac{c}{b}e^{i(\varphi_c-\varphi_b)}. \tag{5}$$

We have that $\varphi_c \approx \pi/4$, or to higher order $\varphi_c \approx \pi/2 + \delta^2\sqrt{R} = \pi/2 + 0.01$. The phase shift on reflection of the Fabry-Perot is $\psi_r = \pi/2 + 0.01 - \pi/4$

### 2.1.3 Difference

Difference of phase shift in reflection and transmission:

$$\psi_r - \psi_t = \frac{\pi}{2} + 0.01 - \pi/4 - 0.05 + \pi/4 \approx \frac{\pi}{2}.$$

Another option is to verify numerically that $\varphi_c = \pi/2 - 0.05$, which gets to the exact verification.

### 2.1.4 Simplest approach

A third option is to note that if $\mathcal{R}$ and $\mathcal{T}$ are $\pi/2$ out of phase, the product $\mathcal{R}^* \times \mathcal{T}$ should be purely imaginary. Indeed, this product is equal to

$$\sqrt{R}[e^{-i\delta/2} - e^{i\delta/2}](1-R)$$

is indeed purely imaginary.

# 3 Resolving the D lines of Sodium

The $D$ lines of sodium have the wavelengths:

$D_1$ 589.756 nm

$D_2$ 589.158 nm

Design a simple Fabry-Perot (index of refraction = 1) to resolve these two lines. The free spectral range of the Fabry-Perot should be twice the difference between these two spectral lines. The second condition is that, to ensure good transmission of either line when at resonance, the FWHM of the Fabry-Perot resonance should be half of the splitting between the lines. The two numbers to give are:

1. What is the spacing between the two reflecting faces of this Fabry-Perot?
2. What is the reflectivity of either face?

## 3.1 Solution

### 3.1.1 Spacing

The free spectral range is

$$\Delta\nu = \frac{1}{\tau_{rt}} = 2(\nu_1 - \nu_2)$$

The thickness (spacing) is thus:

$$d = \frac{c}{4(\nu_1 - \nu_2)} = \frac{0.3}{4 \times 0.515} = 0.145\mathrm{mm}$$

.

### 3.1.2 Reflectivity of either face

The FWHM is twice the half width defined by

$$\frac{R}{(1-R)^2}\delta_{hw}^2 = 1$$

. It is stated that the FWHM should be half of the splitting of the line, which is half the free spectral range (corresponding to $\Delta\delta = 2\pi$). There is no need to "remember a formula": it is easy to deduce the FWHM of a mode of the transmission curve given in Eq. (2). It is acceptable to use that FWHM of the *field* transmission. Therefore:

$$\delta_{hw} = \frac{1-R}{\sqrt{R}} = \frac{2\pi}{8} = \frac{\pi}{4}.$$

The equation to be solved is:

$$R^2 - (2 + \frac{\pi^2}{16})R + 1 = 0.$$

There is only one solution with $R \leq 1$ which is $R = 0.465$.

The "textbook approach" would be to use the FWHM of the *intensity* transmission, in which case we use:

$$\text{Finesse} = \frac{\nu_{\text{freespectralrange}}}{FWHM} = 4 = \frac{\pi R}{1 - R * 2}. \tag{6}$$

which results in the second order equation:

$$4R^2 + \pi R - 4 = 0,$$

which has as solution:

$$R = \frac{-\pi + \sqrt{\pi^2 + 64}}{8} = 0.682.$$