Advanced Optics I (463)
Test 1, October 15, 2013

Application of Fresnel and Snell's equations

Consider an interface between air (index 1) and glass (index n). A beam of width w is incident at an angle θ on that interface. Demonstrate that the incident power equals the reflected power + the transmitted power (energy conservation). Choose s or p polarization (either one should work!).

Solution

Intensity multplied by area ($P = IA$) is what is conserved here, so we must account for the change in beam size on transmission. It can be shown using a purely geometric argument (done several times in class) that the relation between the incident w_i and transmitted w_t beam waist is,

$$w_t = \frac{\cos\theta_t}{\cos\theta_i} w_i. \tag{1}$$

We also know that intensity is proportional to the electric field squared times the index of refraction:

$$I \sim n|\mathcal{E}|^2. \tag{2}$$

If we set the incident electric field amplitude and incident beam area to 1, then our enery conservation equation becomes:

$$P_i = P_r + P_t = I_r A_r + I_t A_t \tag{3}$$

$$1 = n_i |r|^2 + \frac{n_t \cos\theta_t}{\cos\theta_i} |t|^2, \tag{4}$$

where r and t are the fresnel coefficients.

p-polarization

The Fresnel equations give:

$$\begin{aligned} t_\parallel &= \frac{2\cos\theta}{\cos\theta_t + n\cos\theta} \\ r_\parallel &= \frac{\cos\theta_t - n\cos\theta}{\cos\theta_t + n\cos\theta} \end{aligned} \tag{5}$$

Because of the projection of the beam angle on the interface, the energy conservation equation is:

$$R + \frac{n\cos\theta_t}{\cos\theta} T = 1 \tag{6}$$

Substituting:

$$R_\parallel + \frac{n\cos\theta_t}{\cos\theta} T_\parallel = \frac{\cos^2\theta_t + n^2\cos^2\theta - 2n\cos\theta\cos\theta_t + 4n\cos\theta\cos\theta_t}{(\cos\theta_t + n\cos\theta)^2} = 1 \tag{7}$$

s-polarization

The Fresnel equations give:

$$
\begin{aligned}
t_\perp &= \frac{2\cos\theta}{\cos\theta + n\cos\theta_t} \\
r_\perp &= \frac{\cos\theta - n\cos\theta_t}{\cos\theta + n\cos\theta_t}
\end{aligned}
\tag{8}
$$

Because of the projection of the beam angle on the interface, the energy conservation equation is:

$$
R + \frac{n\cos\theta_t}{\cos\theta} T = 1 \tag{9}
$$

Substituting:

$$
R_\perp + \frac{n\cos\theta_t}{\cos\theta} T_\perp = \frac{\cos^2\theta + n^2\cos^2\theta_t - 2n\cos\theta\cos\theta_t + 4n\cos\theta\cos\theta_t}{(\cos\theta + n\cos\theta_t)^2} = 1 \tag{10}
$$

Matrix problem

Several laser cavities are terminated by a curved mirror as output mirror. The rays (inside the laser cavity) are issued from the center of curvature O (see Fig. 1. It is generally desirable that the output beam be collimated. Given a radius of curvature of the first surface $R = 100$ mm, a thickness $t = 8$ mm, calculate within the paraxial approximation, the curvature of the second surface in order to have a collimated output. The index of refraction is $n = 1.5$. Hint: try to solve with the least amount of matrices possible (you do not need to start from point O).

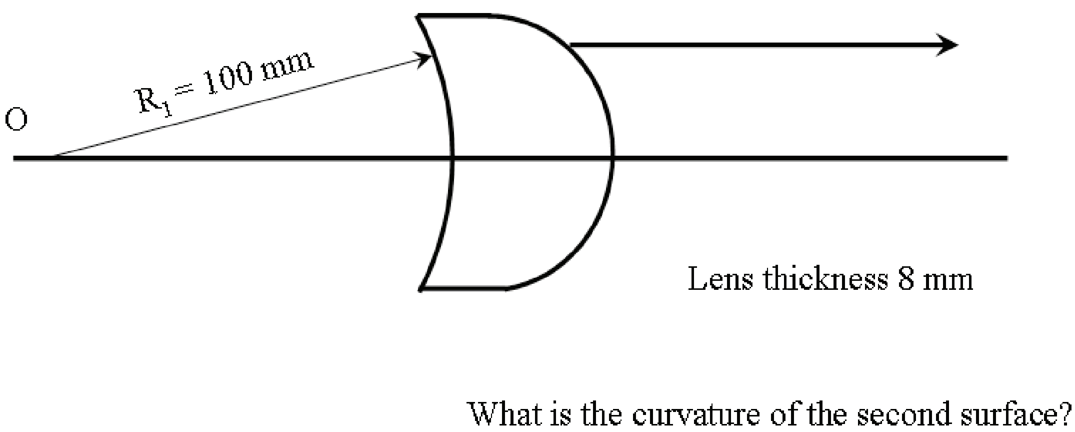

Figure 1:

Solution

In the paraxial approximation ($\sin\alpha \approx \alpha$), the angle that each ray makes with the horizontal axis is $\alpha = y/R_1$ Since any ray issued from O is orthogonal to the spherical input surface of the lens, there is no refraction at the curved surface, and we can use as initial matrix inside the glass:

$$\begin{pmatrix} y \\ \frac{y}{R_1} \end{pmatrix}.$$

All that is left to do is to translate by the thickness t, and the second curved interface:

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{n-1}{R_2} & n \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \frac{y}{R_1} \end{pmatrix} = \begin{pmatrix} 1 & t \\ -\frac{n-1}{R_2} & n - \frac{n-1}{R_2}t \end{pmatrix} \begin{pmatrix} y \\ \frac{y}{R_1} \end{pmatrix}$$

The definition of a collimated beam is that $\alpha' = 0$ which leads to:

$$0 = -(R_1 + t)\frac{n-1}{R_2} + n$$

leading to the result:

$$R_2 = \frac{n-1}{n}(t + R_1) = \frac{0.5}{1.5} \times 108 = 36$$

Prism problem

A beam is incident from the left on the prism sketched in Fig. 2. The index of refraction of the prism is 2.1. It is immersed in a medium of which you vary the index from 1 to infinity.

(a) Complete the figure indicating the path of the beam inside the prism, and the value of the angles of incidence on the various faces.

(b) Assuming the beam is polarized with the electric in the plane of the figure, calculate characteristic values of the index of refraction

(c) Using the characteristic values calculated in (b), sketching (qualitatively) a graph with the values of the intensity transmitted through the face AB versus the index of refraction (*qualitatively*: i.e. it is not necessary to use Fresnel formulae)

(d) What is the value of the intensity transmitted through the face OB, when the outside index takes the values of 1.05 and for 2.1?

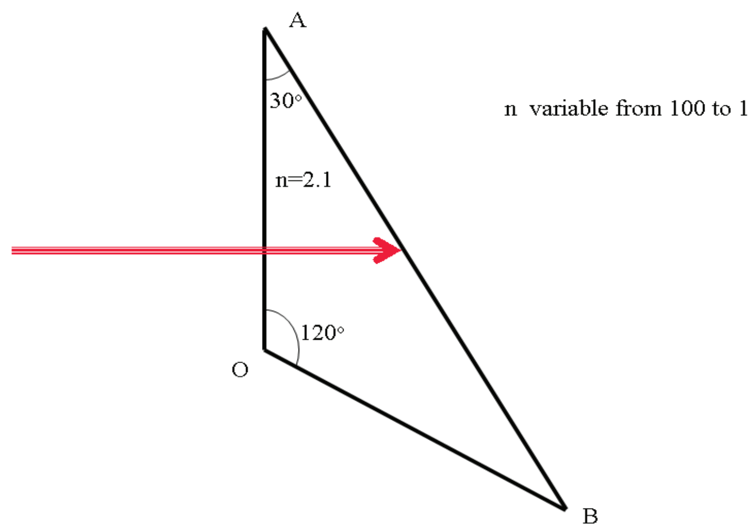

Figure 2:

Solution

a

Because the initial angle of incidence is $0°$, we can conclude that the angle between the input beam and $\overline{AB}$ is $90 - 30 = 60°$. Which means the angle of incidence at the second interface ($\overline{AB}$) must be $30°$. Using the law of reflection ($\theta_i = \theta_r$), we know that the reflected beam will then also have an angle of incidence with the third interface $\overline{OB}$ of $0°$.

b

A beam polarized in the plane of the figure has TM, P, or $\|$ polarization. This means the characteristic values will occur at TIR (n=1.05), Brewster's angle (n=1.21), and when the index of refraction on the outside of the prism is the same as the inside (n=2.1).

c

Below the TIR condition (n=1.05), there will be no power transmitted through interface $\overline{AB}$. After n=1.05, the transmitted power will increase to a local maximum at the Brewster's angle of n=1.21. Note that the transmission will not be 1 here due to the fresnel loss at the first interface. As n keeps increasing the transmitted power will drop slightly before increasing back up to the maximum tranmission of 1 at n=2.1. All power is transmitted here since at n=2.1 there is essentially no prism. As n increases to infinity, the transmitted power will gradually drop due to the fresnel reflections at the interfaces.

d

At n=1.05, the ray will be totally internally reflected at $\overline{AB}$, and therefore the maximum amount of power will be transmitted through $\overline{OB}$. The Intensity will not be 1, however, due to the fresnel reflections at $\overline{AO}$ and $\overline{OB}$. The loss at these interfaces will be,

$$r_{\parallel} = \frac{n_t - n_i}{n_t + n_i} = \frac{2.1 - 1.05}{2.1 + 1.05} = 0.333. \tag{11}$$

Which means the power through interface $\overline{OB}$ when n=1.05 will be 0.778.

When n=2.1, the beam will encounter no interfaces and therefore 0 intensity will be transmitted through $\overline{OB}$.