2–13. Referring to Fig. 2–26, region 2 is free space and the properties of region 1 are $\mu_{R_1} = 1$ and $\epsilon_{R_1} = 9$. The net electric field at the boundary ($d = 0$) is $30\underline{/0}$ V/m. Use the phasor method to calculate $\mathbf{E}_{x_1}$ and $\mathbf{H}_{y_1}$ at $d = 1$ cm and 2 cm if the wave frequency is 1250 MHz. How far from the boundary is the first magnetic field maximum?

2–14. Regions 1 and 2 of Fig. 2–26 contain nonmagnetic dielectrics with $\epsilon_{R_1} = 6$ and $\epsilon_{R_2} = 3$. Calculate the reflection coefficient and SWR for a wave propagating from region 1 toward the dielectric interface.

2–15. A 500 MHz incident wave propagates as shown in Fig. 2–26. Region 1 contains a lossless insulator and region 2 is free space. Determine μ_{R_1} and ϵ_{R_1} if the SWR in region 1 is 1.60 and the wavelength is 35 cm. Assume $\epsilon_{R_1} > \mu_{R_1}$.

2–16. An electromagnetic wave with a power density of 5.0 W/m² impinges on a dielectric boundary causing a SWR of 1.90. Calculate the power density of the wave transmitted into the dielectric.

2–17. Calculate the SWR for the cases of 25 percent and 50 percent power reflected at a dielectric boundary.

2–18. Referring to Fig. 2–29, $\theta_i = 35°$ and region 2 is free space. What is the minimum value of index of refraction for region 1 that results in no transmission into the free space region?

3 Transmission-Line Theory

The theory of electric waves along uniform transmission lines is reviewed in this chapter. A uniform line is defined as one whose dimensions and electrical properties are identical at all planes transverse to the direction of propagation. The analysis includes a study of the reflection characteristics of terminated lines. The results allow us to apply ac circuit concepts to lines whose lengths are *not* negligible compared to the operating wavelength. (The restriction regarding line lengths was discussed in Sec 1–1). An interesting consequence of this analysis is that the impedance of a circuit can be dramatically altered by the addition of a small length of transmission line. This impedance transforming property of a line is a powerful design tool at microwave frequencies. Several illustrative examples are given in this and subsequent chapters.

3–1 CIRCUIT REPRESENTATION OF TRANSMISSION LINES

Transmission lines provide one method of transmitting electrical energy between two points in space, antennas being the other (Appendix F). Figure 1–4 shows four types of lines used at microwave frequencies. The open two-wire line is the most popular at the lower frequencies, the TV twin-lead being a familiar example. UHF and cable TV systems utilize low-loss coaxial cable as a transmission line. Modern microwave practice involves considerable use of coaxial lines at frequencies up to 30 GHz and hollow waveguides from 3 to 300 GHz.

In principle, any transmission line can be analyzed by solving Maxwell's equations and applying the appropriate boundary conditions for the particular line geometry. An example of this is the analysis of hollow waveguides described in Sec. 5–5. A simpler technique that utilizes ac circuit concepts is given in this chapter. As mentioned in Sec. 1–2, this technique was introduced by Lord Kelvin and developed fully by Oliver Heaviside. Essentially, it is an extension of ac circuit theory to lines having distributed circuit elements. A disadvantage of this method is that it reveals little about the electromagnetic field pattern or other possible modes of propagation. However, it does describe the impedance and propagation characteristics of the line for the principal mode of transmission and hence is of considerable engineering value.

The following quantities may be defined for a uniform transmission line.

$R' \equiv$ Series resistance per unit length of line (ohm/m)
$G' \equiv$ Shunt conductance per unit length of line (mho/m)
$L' \equiv$ Series inductance per unit length of line (H/m)
$C' \equiv$ Shunt capacitance per unit length of line (F/m)

The quantity R' is related to the dimensions and conductivity of the metallic conductors. Because of skin effect, it is also a function of frequency. G' is related to the loss tangent of the insulating material between the conductors.[1] L' is associated with the magnetic flux linking the conductors, while C' is associated with the charge on the conductors. Expressions for the distributed elements of various transmission lines are given in Chapter 5.

With this concept of distributed elements, a uniform transmission line may be modeled by the circuit representation in Fig. 3–1. The line is pictured as a cascade of identical sections, each Δz long. Each section consists of series inductance and resistance ($L'\Delta z$ and $R'\Delta z$) as well as shunt capacitance and conductance ($C'\Delta z$ and $G'\Delta z$). Since Δz can always be chosen small compared to the operating wavelength, an individual section of line may be analyzed using ordinary ac circuit theory. In the derivation that follows, $\Delta z \to 0$ and hence the results are valid at all frequencies.

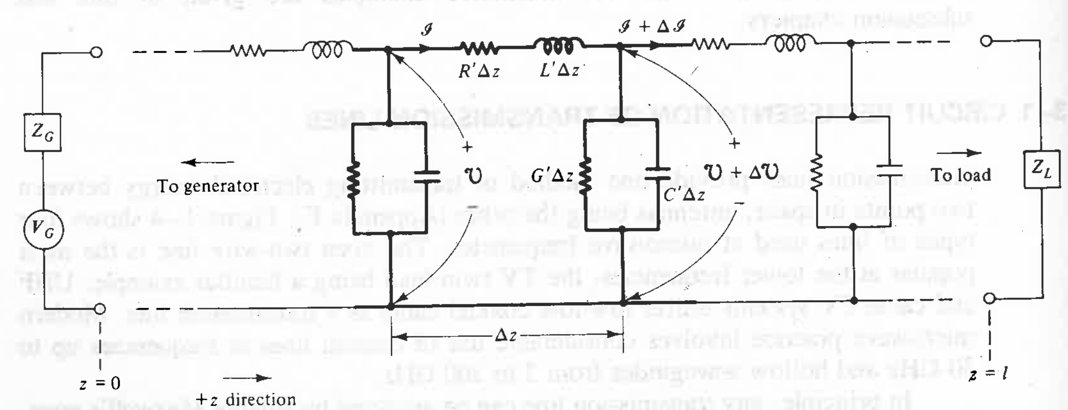

Figure 3–1 Circuit representation of a uniform transmission line.

[1] It is important to note that G' is *not* the reciprocal of R'. They are independent quantities, R' being related to the properties of the two conductors and G' to the characteristics of the insulating material between them.

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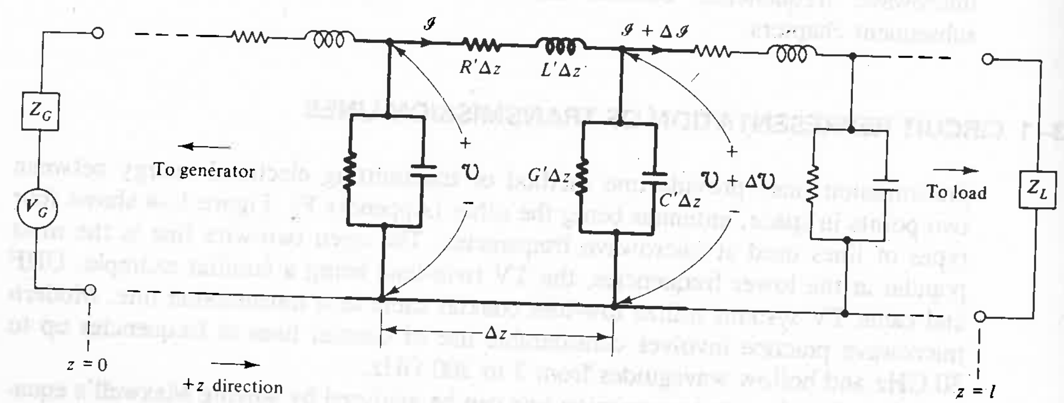

Figure 3–1 Circuit representation of a uniform transmission line.

[1] It is important to note that G' is *not* the reciprocal of R'. They are independent quantities, R' being related to the properties of the two conductors and G' to the characteristics of the insulating material between them.

In the figure, $\mathcal{V}$ and $\mathcal{I}$ represent the time-varying voltage and current at the input of a line section, while $\mathcal{V} + \Delta\mathcal{V}$ and $\mathcal{I} + \Delta\mathcal{I}$ represent the output values. The positive z direction is taken as horizontal and to the right, that is, from the generator toward the load. Also indicated are the assumed positive directions for the currents and voltages.

Applying Kirchhoff's voltage and current laws to the line section yields

$$\mathcal{V} = (R'\Delta z)\mathcal{I} + (L'\Delta z)\frac{\partial \mathcal{I}}{\partial t} + (\mathcal{V} + \Delta\mathcal{V})$$

and

$$\mathcal{I} = (G'\Delta z)(\mathcal{V} + \Delta\mathcal{V}) + (C'\Delta z)\frac{\partial}{\partial t}(\mathcal{V} + \Delta\mathcal{V}) + (\mathcal{I} + \Delta\mathcal{I})$$

Simplifying and recognizing that as $\Delta z \to 0$, $\mathcal{V} + \Delta\mathcal{V} \to \mathcal{V}$ results in the following partial differential equations.

$$-\frac{\partial \mathcal{V}}{\partial z} = R'\mathcal{I} + L'\frac{\partial \mathcal{I}}{\partial t} \quad \text{and} \quad -\frac{\partial \mathcal{I}}{\partial z} = G'\mathcal{V} + C'\frac{\partial \mathcal{V}}{\partial t} \tag{3–1}$$

By taking $\partial/\partial z$ of the first equation and $\partial/\partial t$ of the second equation and eliminating $\partial \mathcal{I}/\partial z$ and $\partial^2 \mathcal{I}/\partial z\,\partial t$, a second-order differential equation for voltage is obtained.

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{V}}{\partial t^2} + (R'C' + G'L')\frac{\partial \mathcal{V}}{\partial t} + R'G'\mathcal{V} \tag{3–2}$$

Solving for current in a similar manner yields

$$\frac{\partial^2 \mathcal{I}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{I}}{\partial t^2} + (R'C' + G'L')\frac{\partial \mathcal{I}}{\partial t} + R'G'\mathcal{I} \tag{3–3}$$

The solution of either of these second-order equations and Eq. (3–1), together with the electrical properties of the generator and load, allow us to determine the instantaneous voltage and current at any time t and any place z along the uniform transmission line.

For the case of perfect conductors ($R' = 0$) and insulators ($G' = 0$), the above equations reduce to

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{V}}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 \mathcal{I}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{I}}{\partial t^2} \tag{3–4}$$

while Eqs. (3–1) reduce to

$$-\frac{\partial \mathcal{V}}{\partial z} = L'\frac{\partial \mathcal{I}}{\partial t} \quad \text{and} \quad -\frac{\partial \mathcal{I}}{\partial z} = C'\frac{\partial \mathcal{V}}{\partial t} \tag{3–5}$$

Equations (3–4) and (3–5) represent the differential equations for a lossless line. Although real lines are never without loss, there are many in which it is sufficiently small that the lossless solution represents an excellent approximation.

Equations (3–4) are forms of the well-known wave equation of mathematical physics. We have already encountered it in Eq. (2–45). It was shown that the solution represented electromagnetic waves traveling in the plus and minus z directions with a velocity given by Eq. (2–53). The solutions of Eqs. (3–4) also represent traveling waves. In this case, they are voltage and current waves that travel with a velocity given by

$$v = \frac{1}{\sqrt{L'C'}} \tag{3–6}$$

In general, Eqs. (3–4) are satisfied by single-valued functions of the form $f(t \pm \sqrt{L'C'}\,z)$, where the plus sign indicates propagation in the *negative* z direction and the minus sign propagation in the *positive z* direction. To understand the meaning of these solutions, assume $\mathcal{V} = f(t - \sqrt{L'C'}\,z)$. At the point $z = 0$, the voltage versus time function is given by $\mathcal{V} = f(t)$. Further down the z axis at a point $z = z_1$, $\mathcal{V} = f(t - \sqrt{L'C'}\,z_1)$, which is exactly the same as $f(t)$ except that it has been time delayed by $t_d = \sqrt{L'C'}\,z_1$. Thus, it appears that the voltage versus time function at $z = 0$ has moved to $z = z_1$ with a velocity $v = z_1/t_d = 1/\sqrt{L'C'}$, which is exactly Eq. (3–6). By a similar argument, the $f(t + \sqrt{L'C'}\,z)$ solution represents a voltage function traveling in the negative z direction. In like manner, the solutions of the current equation may be interpreted as forward and reverse traveling current functions having the same velocity as the voltage. A similar conclusion was arrived at regarding the $\mathscr{E}$ and $\mathscr{H}$ waves discussed in Sec. 2–4, the explanation being that $\mathscr{E}$ generated $\mathscr{H}$ and vice versa. The voltage and current waves also travel with the same velocity since $\mathcal{V}$ and $\mathcal{I}$ generate each other. A physical explanation is presented in the next section to show the reasonableness of this conclusion.

Another result given in Sec. 2–4 is that the ratio of $\mathscr{E}$ to $\mathscr{H}$ for the traveling waves is a constant (η) which is a function of the electric and magnetic properties of the medium. Similarly, the ratio of $\mathcal{V}$ to $\mathcal{I}$ for a traveling wave on a transmission line is a constant. This constant is called the *characteristic impedance* (Z_0) of the line. For a lossless line, it is given by

$$Z_0 = \sqrt{\frac{L'}{C'}} \quad \text{ohms} \tag{3–7}$$

To verify this expression, let $\mathcal{V} = f_1(u)$ and $\mathcal{I} = f_2(u)$, where $u \equiv t - \sqrt{L'C'}\,z$. Since

$$\frac{\partial \mathcal{V}}{\partial z} = \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial z} = -\sqrt{L'C'}\frac{\partial f_1}{\partial u} \qquad \text{and} \qquad \frac{\partial \mathcal{I}}{\partial t} = \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial t} = \frac{\partial f_2}{\partial u}$$

substitution into the first of Eqs. (3–5) yields $\sqrt{L'C'}\,\partial f_1/\partial u = L'\partial f_2/\partial u$. Integration with respect to u and simplifying results in $f_1/f_2 = \mathcal{V}/\mathcal{I} = \sqrt{L'/C'}$, which is Eq. (3–7).

It will be shown that Z_0 is a function of the cross-sectional dimensions of the line as well as the electrical properties of the insulating material between the conductors.

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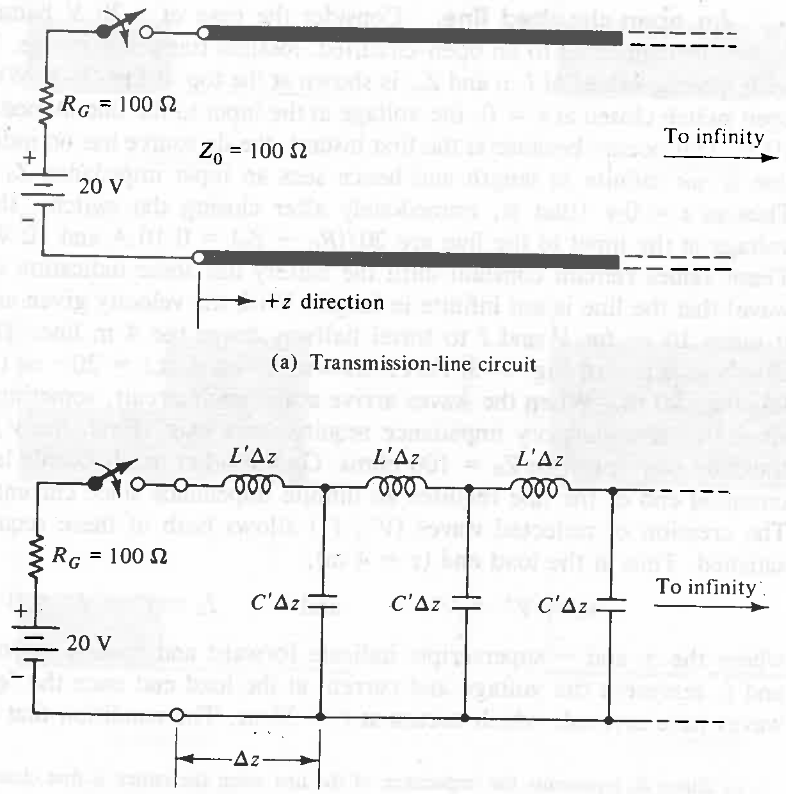
It will be shown that Z_0 is a function of the cross-sectional dimensions of the line as well as the electrical properties of the insulating material between the conductors.

3–2 TRANSIENTS ON A TRANSMISSION LINE

We have seen that voltage and current waves travel along a transmission line with the same velocity. The physical argument presented here is intended to verify this fact while giving additional insight into the process of wave propagation along uniform lines.

Figure 3-2a shows a 20 V battery with an internal resistance of 100 ohms connected through a switch to an infinitely long transmission line with $Z_0 = 100$ ohms. Part *b* of the figure shows the same circuit with the transmission line replaced by its equivalent circuit representation. When the switch is closed, a voltage appears immediately at the input of the transmission line. However, it cannot appear instantaneously at other points along the line, since that would require a sudden change in voltage on all the capacitances. Furthermore, since it is current that delivers the electric charge to the capacitances, a sudden increase in current through the inductances would also be necessary. Since inductance opposes a current change and capacitance opposes a voltage change, the voltage and current require a finite time to propagate along the transmission line. The propagation process can be described in



(a) Transmission-line circuit

(b) Equivalent circuit representation

Figure 3–2 A dc source connected to an infinitely long, lossless line.

the following manner. When the switch is closed, the first inductance generates a back emf, in accordance with Lenz's law, to initially oppose an increase in current. Eventually, however, current flows through $L'\Delta z$ and charges the first shunt capacitance $C'\Delta z$ to a voltage V. The charged capacitor now acts like a voltage source and forces current through the next inductor. This charges the next capacitor and the process continues down the line. From this argument, it is apparent that voltage creates current and vice versa, thus requiring that they travel together along the transmission line. Since the line is infinitely long, only forward traveling waves of voltage and current exist and their ratio is given by Z_0 (100 ohms in this case). As time progresses, the battery continues to supply the current needed to charge the never-ending line of shunt capacitances. Thus, in the steady state, the infinite line presents an impedance of Z_0 to the battery. The current supplied by the battery is $20/(R_G + Z_0) = 0.10$ A. With half of the 20 V dropped across R_G, the voltage at the input to the line is 10 V. This 10 V voltage wave and its accompanying 0.10 A current wave travel in the positive z direction with a velocity given by Eq. (3–6).

Let us now look at some examples of finite length lines with various terminations.

An open-circuited line. Consider the case of a 20 V battery with $R_G =$ 100 ohms connected to an open-circuited, lossless transmission line. This situation, with specific values of l, v and Z_0, is shown at the top of Fig. 3–3. With the initially open switch closed at $t = 0$, the voltage at the input to the line immediately becomes 10 V. This occurs because at the first instant, the dc source has no indication that the line is *not* infinite in length and hence sees an input impedance $Z_0 = 100$ ohms.[2] Thus at $t = 0+$ (that is, immediately after closing the switch), the current and voltage at the input to the line are $20/(R_G + Z_0) = 0.10$ A and 10 V, respectively. These values remain constant until the battery has some indication (via a reflected wave) that the line is not infinite in length. With the velocity given as 2×10^8 m/s, it takes 10 ns for V and I to travel halfway down the 4 m line. This situation is shown in part *a* of Fig. 3–3. Part *b* shows the waves at $t = 20-$ ns (that is, slightly less than 20 ns). When the waves arrive at the open circuit, something must happen since two contradictory impedance requirements exist. First, the V/I ratio for the traveling wave must be $Z_0 = 100$ ohms. On the other hand, Ohm's law at the open-circuited end of the line requires an infinite impedance since current must be zero. The creation of reflected waves (V^-, I^-) allows both of these requirements to be satisfied. Thus at the load end ($z = 4$ m),

$$V_L = V^+ + V^- \qquad \text{and} \qquad I_L = I^+ - I^- = 0$$

where the + and − superscripts indicate forward and reverse traveling waves.[3] V_L and I_L represent the voltage and current at the load end once the forward traveling waves have arrived, which occurs at $t = 20$ ns. The condition that $I_L = 0$ requires

[2] Since Z_0 represents the impedance of the line when the switch is first closed, it is sometimes referred to as the *surge impedance* of the transmission line.

[3] The reason for the minus sign in the equation for I_L is that for the forward and reverse traveling waves, I^+ and I^- are oppositely directed.

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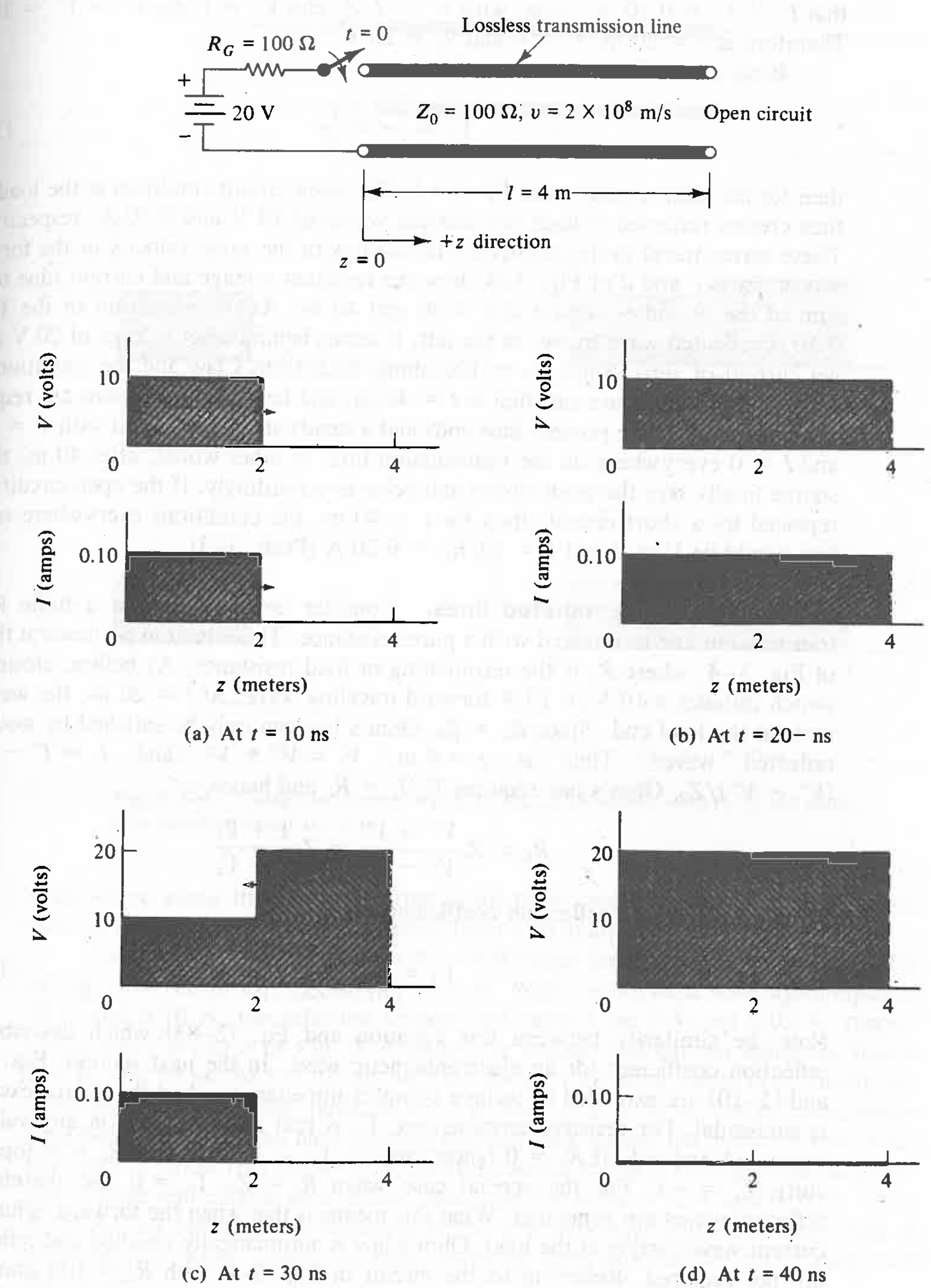

Figure 3–3 Voltage and current waves on an open-circuited transmission line. (The switch is closed at t = 0.)

that $I^- = I^+ = 0.10$ A. Also, with $V^+ = I^+Z_0$ and $V^- = I^-Z_0$, $V^- = V^+ = 10$ V. Therefore at $t = 20$ ns, $I_L = 0$ and $V_L = 20$ V.

If we define the reflection coefficient at the load as

$$\Gamma_L \equiv \frac{V^-}{V^+} = \frac{I^-}{I^+} \tag{3-8}$$

then for the open-circuit case, $\Gamma_L = +1$. The open-circuit condition at the load end thus creates reflected voltage and current waves of 10 V and 0.10 A, respectively. These waves travel in the negative z direction with the same velocity as the forward waves. Parts c and d of Fig. 3–3 show the resultant voltage and current (due to the sum of the + and − waves) at $t = 30$ and 40 ns. As the wavefront of the 10 V, 0.10 A reflected wave moves to the left, it leaves behind a net voltage of 20 V and a net current of zero. Since $R_G = 100$ ohms, both Ohm's law and the condition that $V^-/I^- = 100$ ohms are satisfied at $t = 40$ ns, and hence no reflections are required at the generator. The process thus ends and a steady state is achieved with $V = 20$ V and $I = 0$ everywhere on the transmission line. In other words, after 40 ns, the dc source finally *sees* the open circuit and behaves accordingly. If the open circuit were replaced by a short circuit, then for $t > 40$ ns, the conditions everywhere on the line would be $V = 0$ and $I = 20/R_G = 0.20$ A (Prob. 3–3).

Resistively terminated lines. Consider now the case of a finite length transmission line terminated with a pure resistance. This situation is shown at the top of Fig. 3–4, where R_L is the terminating or load resistance. As before, closing the switch initiates a 10 V, 0.10 A forward traveling wave. At $t = 20$ ns, the wave arrives at the load end. Since $R_L \neq Z_0$, Ohm's law can only be satisfied by assuming reflected waves. Thus at $z = 4$ m, $V_L = V^+ + V^-$ and $I_L = I^+ - I^- = (V^+ - V^-)/Z_0$. Ohm's law requires $V_L/I_L = R_L$ and hence

$$R_L = Z_0\frac{V^+ + V^-}{V^+ - V^-} = Z_0\frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{3-9}$$

Solving for the load reflection coefficient yields

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{3-10}$$

Note the similarity between this equation and Eq. (2–83) which describes the reflection coefficient for an electromagnetic wave. In the next section, Eqs. (3–9) and (3–10) are extended to include complex impedances when the source excitation is sinusoidal. For resistive terminations, Γ_L is real and can take on any value between −1 and +1. If $R_L = 0$ (short circuit), $\Gamma_L = -1$, while if $R_L = \infty$ (open circuit), $\Gamma_L = +1$. For the special case when $R_L = Z_0$, $\Gamma_L = 0$ and therefore no reflected waves are generated. What this means is that when the forward voltage and current waves arrive at the load, Ohm's law is automatically satisfied and reflections are not required. Referring to the circuit in Fig. 3–4 with $R_L = 100$ ohms, the steady-state condition is reached after 20 ns, namely, $V = 10$ V and $I = 0.10$ A that $I^- = I^+ = 0.10$ A. Also, with $V^+ = I^+Z_0$ and $V^- = I^-Z_0$, $V^- = V^+ = 10$ V. Therefore at $t = 20$ ns, $I_L = 0$ and $V_L = 20$ V.

If we define the reflection coefficient at the load as

$$\Gamma_L \equiv \frac{V^-}{V^+} = \frac{I^-}{I^+} \tag{3-8}$$

then for the open-circuit case, $\Gamma_L = +1$. The open-circuit condition at the load end thus creates reflected voltage and current waves of 10 V and 0.10 A, respectively. These waves travel in the negative z direction with the same velocity as the forward waves. Parts c and d of Fig. 3–3 show the resultant voltage and current (due to the sum of the + and − waves) at $t = 30$ and 40 ns. As the wavefront of the 10 V, 0.10 A reflected wave moves to the left, it leaves behind a net voltage of 20 V and a net current of zero. Since $R_G = 100$ ohms, both Ohm's law and the condition that $V^-/I^- = 100$ ohms are satisfied at $t = 40$ ns, and hence no reflections are required at the generator. The process thus ends and a steady state is achieved with $V = 20$ V and $I = 0$ everywhere on the transmission line. In other words, after 40 ns, the dc source finally *sees* the open circuit and behaves accordingly. If the open circuit were replaced by a short circuit, then for $t > 40$ ns, the conditions everywhere on the line would be $V = 0$ and $I = 20/R_G = 0.20$ A (Prob. 3–3).

Resistively terminated lines. Consider now the case of a finite length transmission line terminated with a pure resistance. This situation is shown at the top of Fig. 3–4, where R_L is the terminating or load resistance. As before, closing the switch initiates a 10 V, 0.10 A forward traveling wave. At $t = 20$ ns, the wave arrives at the load end. Since $R_L \neq Z_0$, Ohm's law can only be satisfied by assuming reflected waves. Thus at $z = 4$ m, $V_L = V^+ + V^-$ and $I_L = I^+ - I^- = (V^+ - V^-)/Z_0$. Ohm's law requires $V_L/I_L = R_L$ and hence

$$R_L = Z_0\frac{V^+ + V^-}{V^+ - V^-} = Z_0\frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{3-9}$$

Solving for the load reflection coefficient yields

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{3-10}$$

Note the similarity between this equation and Eq. (2–83) which describes the reflection coefficient for an electromagnetic wave. In the next section, Eqs. (3–9) and (3–10) are extended to include complex impedances when the source excitation is sinusoidal. For resistive terminations, Γ_L is real and can take on any value between −1 and +1. If $R_L = 0$ (short circuit), $\Gamma_L = -1$, while if $R_L = \infty$ (open circuit), $\Gamma_L = +1$. For the special case when $R_L = Z_0$, $\Gamma_L = 0$ and therefore no reflected waves are generated. What this means is that when the forward voltage and current waves arrive at the load, Ohm's law is automatically satisfied and reflections are not required. Referring to the circuit in Fig. 3–4 with $R_L = 100$ ohms, the steady-state condition is reached after 20 ns, namely, $V = 10$ V and $I = 0.10$ A

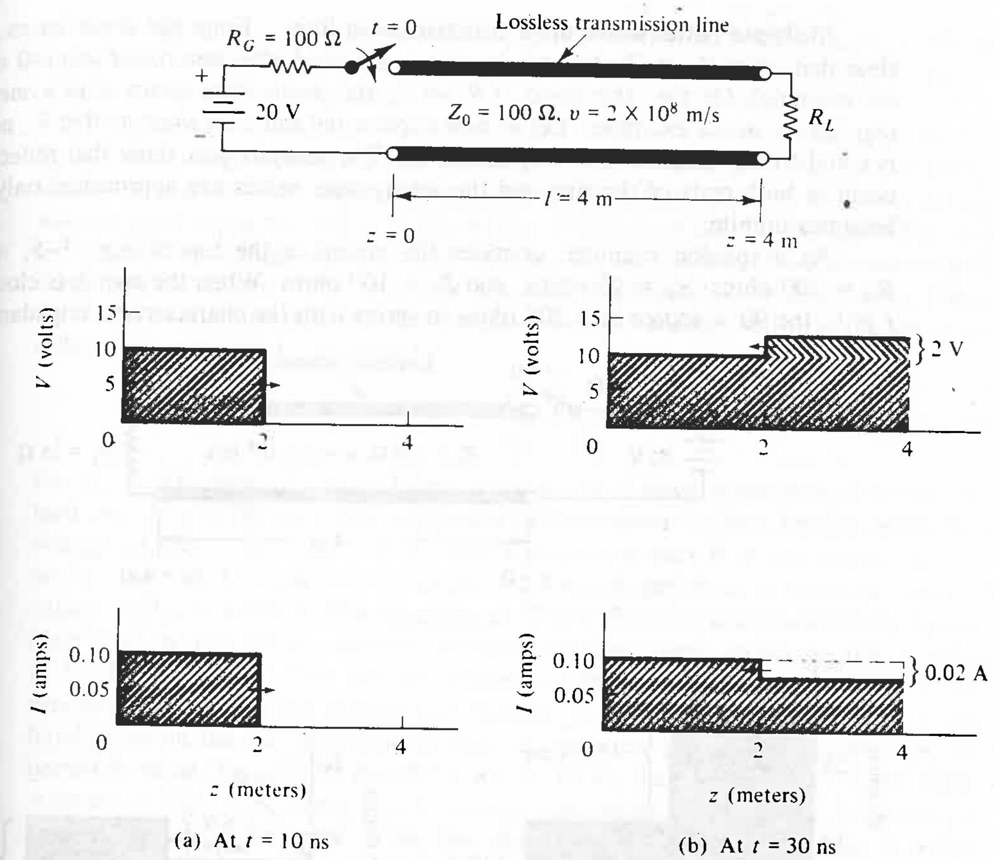

Figure 3–4 Voltage and current waves for the circuit shown when $R_L = 150$ ohms. (The switch is closed at $t = 0$.)

everywhere along the line. Note that in all these cases, the steady-state values of voltage and current are those expected from a dc analysis of the circuit.

Consider now the case where $R_L = 150$ ohms for the circuit shown at the top of Fig. 3–4. From Eq. (3–10), $\Gamma_L = 0.20$. With the forward wave again equal to 10 V and 0.10 A, the reflected voltage and current are 2 V and 0.02 A, respectively. Parts a and b of Fig. 3–4 shows the voltage and current along the line at $t = 10$ and 30 ns. At $t = 10$ ns, only the forward traveling waves exist, having arrived only at the halfway point of the 4 m line. At $t = 30$ ns, the reflected waves have been generated and have traveled halfway back toward the generator end of the line. At $t = 40$ ns (not shown), the reflected waves arrive at the input and the resultant voltage and current everywhere along the line become 12 V and 0.08 A. Since $R_G = Z_0$, no reflection is required at the generator end and the steady state is achieved after 40 ns. Again, the final values are those expected from a dc analysis of the circuit.

Multiple reflections on a transmission line. From the above cases, it is clear that when $R_G = Z_0$, the steady state is achieved after one *round trip* (40 ns, in our example). On the other hand, if $R_L = Z_0$, the steady state occurs after a *one-way trip* (20 ns, in our example). Let us now explore the situation when neither R_G nor R_L is equal to the characteristic impedance Z_0. The analysis will show that reflections occur at both ends of the line and the steady-state values are approached only as t becomes infinite.

As a specific example, consider the circuit at the top of Fig. 3–5, where $R_G = 200$ ohms, $R_L = 25$ ohms, and $Z_0 = 100$ ohms. When the switch is closed at $t = 0$, the 90 V source sees 200 ohms in series with the characteristic impedance of

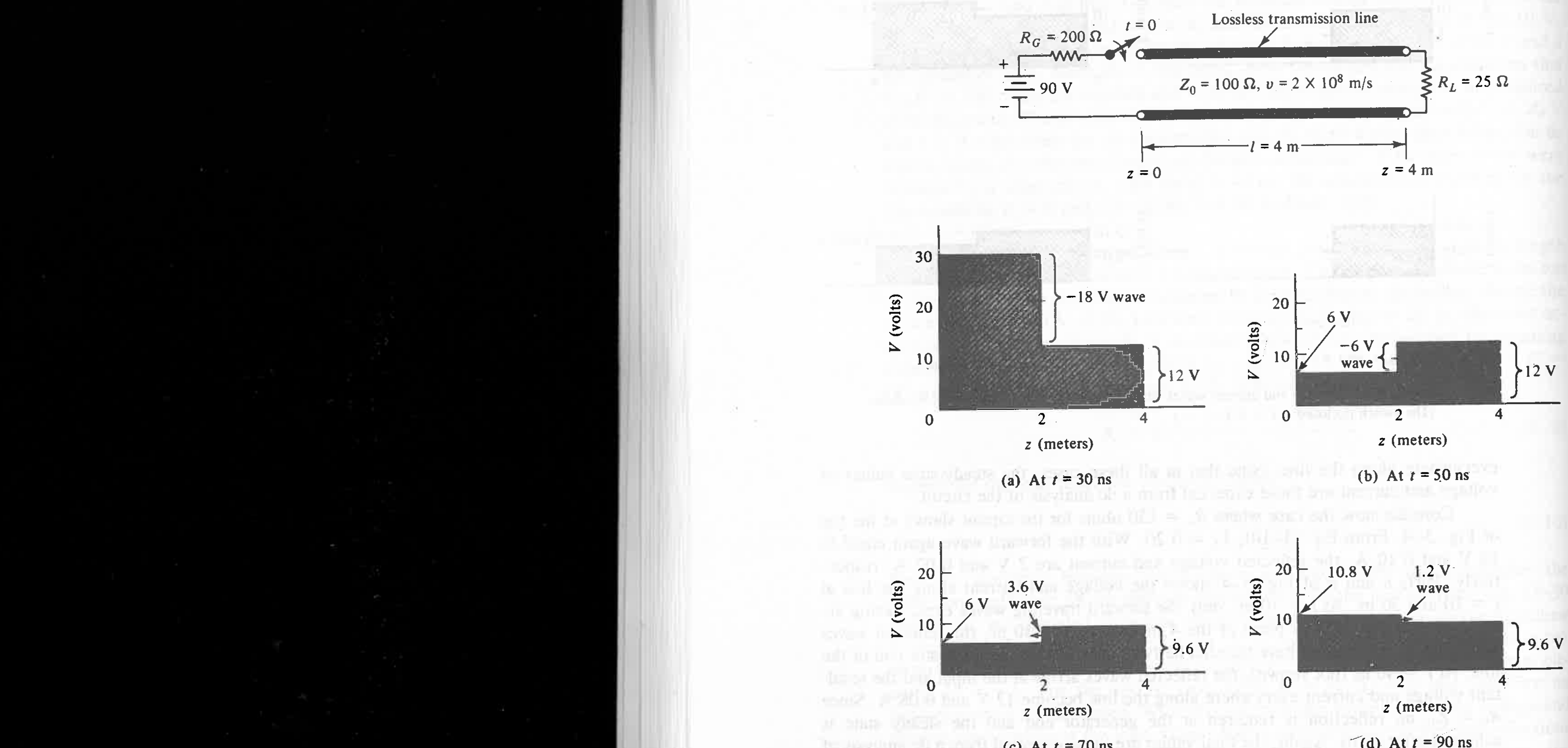

Figure 3–5 Multiple reflections on a resistively terminated transmission line. Z_0 = 100 ohms, R_G = 200 ohms, and R_L = 25 ohms. (The current waves are not shown.)

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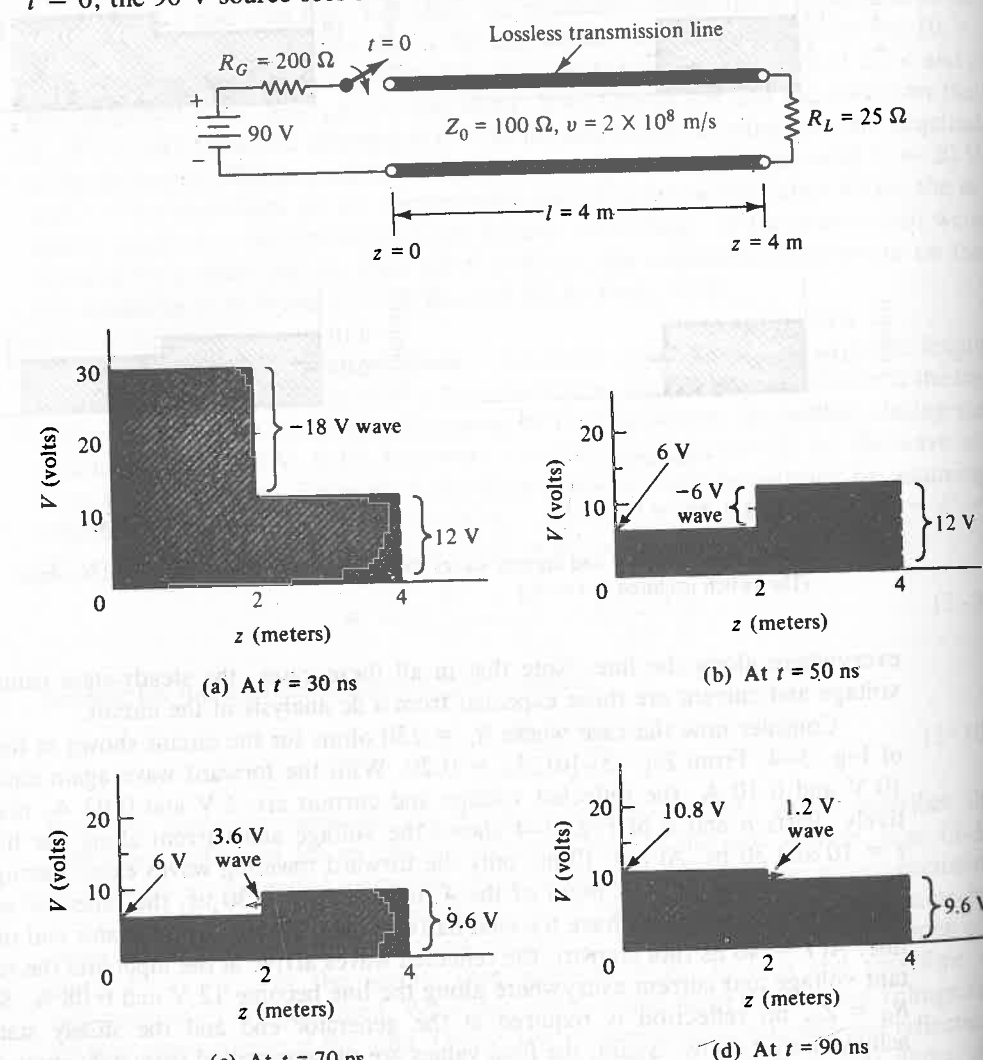


Figure 3–5 Multiple reflections on a resistively terminated transmission line. $Z_0 = 100$ ohms, $R_G = 200$ ohms, and $R_L = 25$ ohms. (The current waves are not shown.)

the line. Therefore, the current and voltage at the input end of the line ($z = 0$) are initially $I^+ = 90/300 = 0.3$ A and $V^+ = I^+ Z_0 = 30$ V. After 20 ns, the V^+ and I^+ waves arrive at the load end where the reflection coefficient $\Gamma_L = -75/125 = -0.6$ and hence $V^- = \Gamma_L V^+ = -18$ V and $I^- = \Gamma_L I^+ = -0.18$ A. At the end of 30 ns, the voltage between $z = 2$ m and $z = 4$ m is reduced to $30 - 18 = 12$ V, while the current has increased to $0.3 + 0.18 = 0.48$ A. The progress of the voltage wave along the line is shown in Fig. 3–5 for $t = 30$, 50, 70, and 90 ns. Let us observe the voltage wave as time marches on. At the end of 40 ns, the -18 V wave arrives at the input where it sees an impedance $R_G = 200$ ohms. Since $R_G \neq Z_0$, a reflection occurs at the generator end. By analogy with Γ_L, the generator reflection coefficient Γ_G is given by

$$\Gamma_G = \frac{R_G - Z_0}{R_G + Z_0} \tag{3–11}$$

For $R_G = 200$ ohms, $\Gamma_G = 1/3$ and hence a -6 V wave is rereflected toward the load end. At $t = 50$ ns, it has progressed halfway down the line, leaving behind it a voltage of $(30 - 18 - 6) = 6$ V. This is shown in part *b* of the figure. At $t = 60$ ns, the -6 V wave arrives at the load which generates a reflected wave of value $(-6)\Gamma_L = +3.6$ V. The situations at 70 and 90 ns are also shown in the figure. Note that at $t = 90$ ns, another forward traveling wave exists having a value $(+3.6)\Gamma_G = +1.2$ V. This process continues indefinitely with the amplitude of the rereflected waves getting smaller and smaller. A plot of voltage versus time at any fixed point on the line would show that, in the limit, the voltage becomes the expected dc value (namely, $90\ R_L/(R_G + R_L) = 10$ V). Such a plot at $z = 0$, the input, is shown in Fig. 3–6. Every step in voltage represents the arrival and generation of reflected waves at the input. After five *round trips* (200 ns), the voltage is within 0.10 percent of the steady-state value.

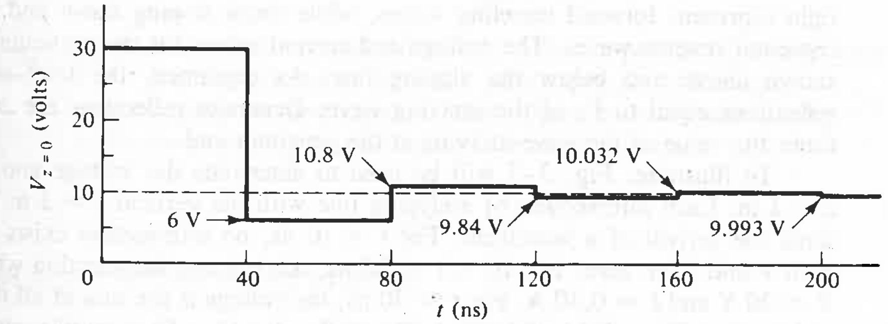


Figure 3–6 Input voltage versus time for the line shown in Fig. 3–5.

The space-time diagram developed by Bewley (Ref. 3–6) is a graphic aid in determining the voltage and current as a function of either time or position along the line. Figure 3–7 shows the diagram for the circuit conditions in Fig. 3–5. The abscissa indicates position along the line and the ordinate represents the time scale, $t = 0$ being the moment that the switch is closed. For reference, the values of Γ_G and Γ_L are given at the top of the diagram. The lines sloping downward and to the

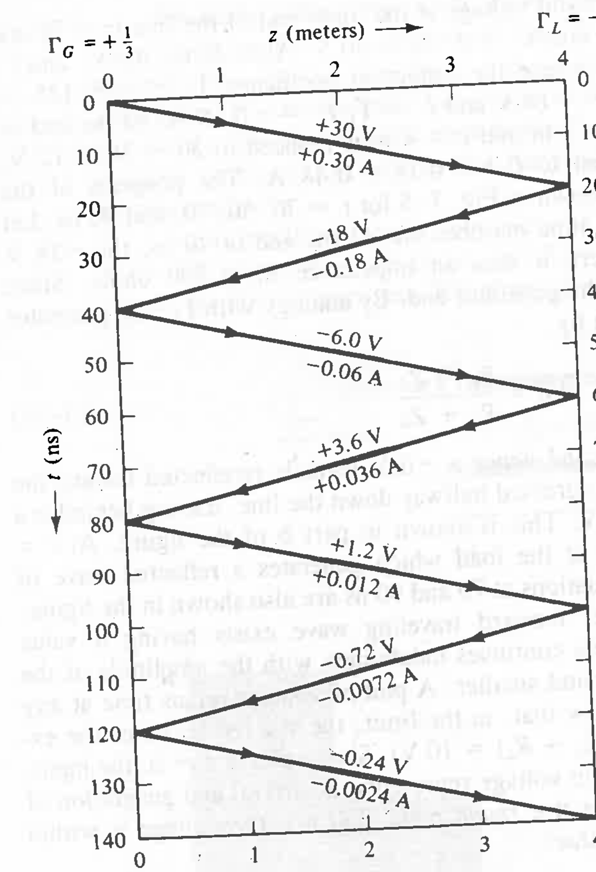
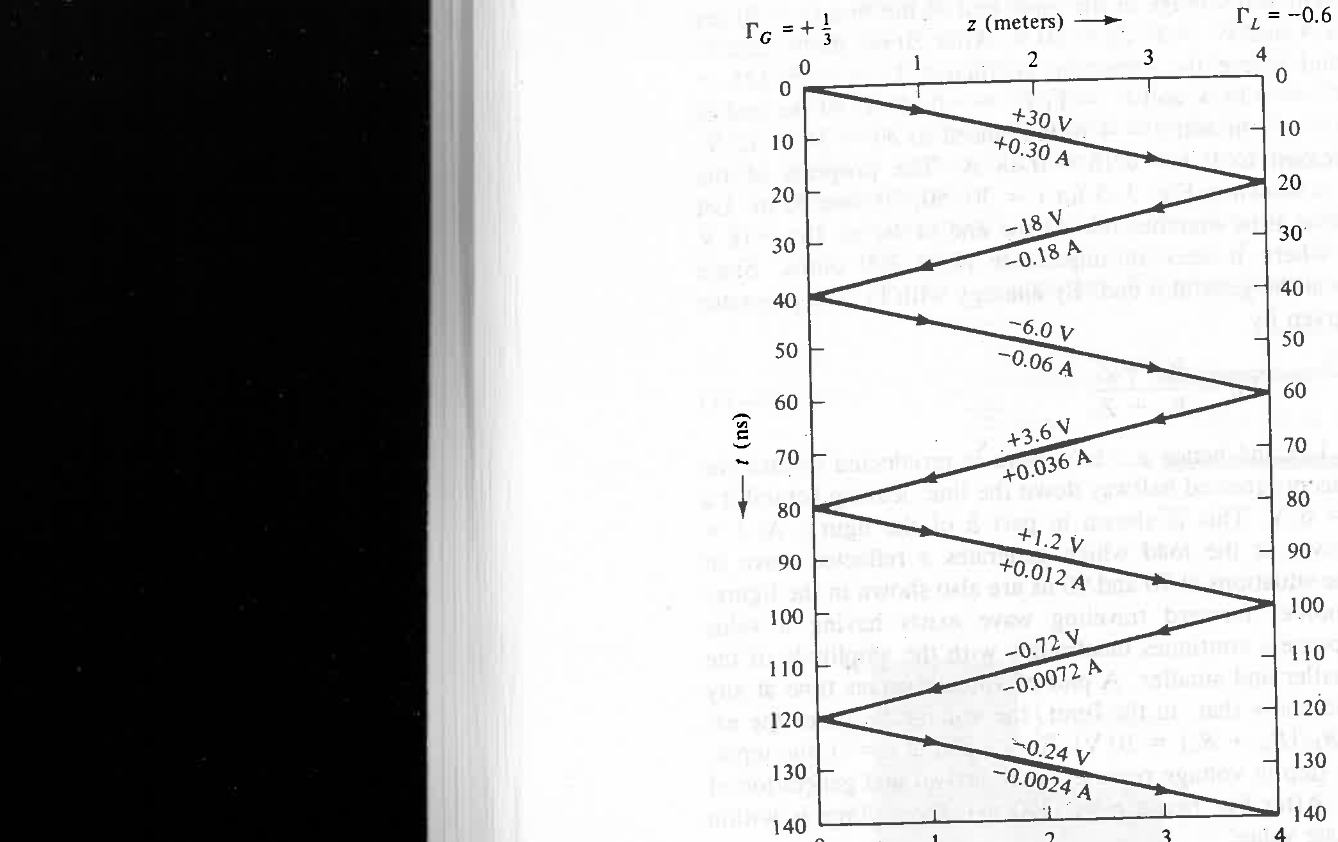


Figure 3–7 Space-time diagram for the transmission-line circuit shown in Fig. 3–5.

right represent forward traveling waves, while those sloping down and to the left represent reverse waves. The voltage and current values for the particular wave are shown above and below the sloping line. As explained, the load end creates reflections equal to Γ_L of the arriving wave. Generator reflections are equal to Γ_G times the value of the wave arriving at the generator end.

To illustrate, Fig. 3–7 will be used to determine the voltage and current at $z = 2$ m. Each intersection of a sloping line with the vertical $z = 2$ m line represents the arrival of a wavefront. For $t < 10$ ns, no intersection exists and hence both V and I are zero. For $10 < t < 30$ ns, there is one intersection which means $V = 30$ V and $I = 0.30$ A. For $t > 30$ ns, the voltage is the sum of all the forward and reverse waves that have passed the $z = 2$ m location. For example, at $t = 80$ ns, $V = 30 - 18 - 6 + 3.6 = 9.6$ V. The current may be determined in a similar manner except that current values associated with reverse waves must be subtracted from those associated with the forward waves. For example, at $t = 80$ ns, $I = 0.30 - (-0.18) + (-0.06) - (+0.036) = 0.384$ A. The diagram may also be used to determine voltage and current versus z for a fixed time by drawing a horizontal line corresponding to the particular value of time. The sum of the voltages above the line correspond to the voltage at that point on the line. The same applies



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to the current except that, as before, reverse-traveling current waves must be subtracted from forward-traveling current waves.

The space-time diagram may be extended to transmission lines having discontinuities and branches (Chapter 3 of Ref. 3–7).

It is interesting to note that the voltage shown in Fig. 3–6 is oscillatory as it approaches its final value. The period of this *ringing* effect is 80 ns (twice the *round-trip* time) and hence its reciprocal is the natural resonant frequency of the circuit, namely, 12.5 MHz. Since $v = 2 \times 10^8$ m/s, this means that the line is $\lambda/4$ long at the resonant frequency. Thus we see that by connecting a dc source to a transmission line, high frequency oscillations are possible. Granted, the oscillation is heavily damped in this example, but the damping can be reduced by increasing the magnitude of both reflection coefficients. In fact, if they are both unity, the oscillation will continue indefinitely (Prob. 3–6). In other words, a configuration consisting of two large reflections separated by a length of transmission line has the properties of a resonant circuit. Most of the microwave resonators described in Chapter 9 have exactly this configuration.

3–3 SINUSOIDAL EXCITATION OF TRANSMISSION LINES

Let us now turn to the important case of uniform transmission lines with sinusoidal excitation. Since our interest is in the steady-state solution, the rms-phasor method, reviewed in Sec. 1–4, will be employed.

Equations (3–1) resulted from a distributed circuit analysis of the uniform transmission line described in Fig. 3–1. Written in phasor form, they become

$$-\frac{d\mathbf{V}}{dz} = (R' + j\omega L')\mathbf{I} = Z'\mathbf{I} \quad \text{and} \quad -\frac{d\mathbf{I}}{dz} = (G' + j\omega C')\mathbf{V} = Y'\mathbf{V} \qquad (3\text{–}12)$$

where $Z' \equiv R' + j\omega L'$ is defined as the series impedance per unit length and $Y' \equiv G' + j\omega C'$ is defined as the shunt admittance per unit length.

Differentiating the first equation with respect to z and substituting $-Y'\mathbf{V}$ for $d\mathbf{I}/dz$ yields the following second-order differential equation.

$$\frac{d^2\mathbf{V}}{dz^2} = Z'Y'\mathbf{V} \qquad (3\text{–}13)$$

Its phasor solution may be written as

$$\mathbf{V} = \mathbf{V}_0^+ e^{-\gamma z} + \mathbf{V}_0^- e^{+\gamma z} = \mathbf{V}^+ + \mathbf{V}^- \qquad (3\text{–}14)$$

where γ is the propagation constant and given by

$$\gamma = \sqrt{Z'Y'} = \sqrt{(R' + j\omega L')(G' + j\omega C')} \qquad (3\text{–}15)$$

In general, γ is complex and may be written as $\gamma = \alpha + j\beta$, where as explained in Sec. 2–6 α is the attenuation constant (Np/length) and β is the phase constant (rad/length).