# Coherent Interactions - just 2 levels

We consider a laser pulse described by:

$$E(t) = \mathcal{E}_1(t)e^{i[\omega_{\ell,1}t+\varphi_1(t)]} + c.c. \tag{1}$$

The relevant two level system is sketched in Fig. 1. The detuning is defined as:

$$\Delta_1 = \omega_{01} - \omega_{\ell,1} \tag{2}$$

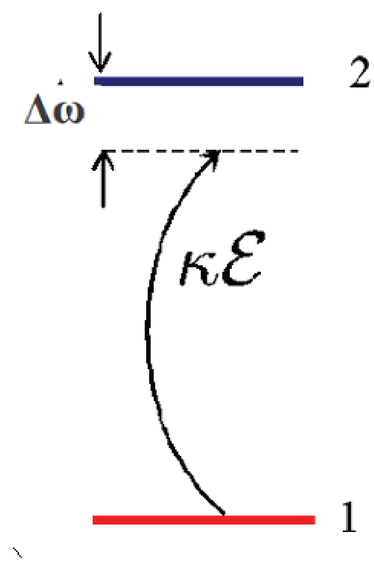


Figure 1: Two-level system.

The coupling with the system is through the dipole interaction term in the time dependent Schrödinger equation:

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{3}$$

with:

$$H = H_0 + H' = H_0 - p \cdot E(t) \tag{4}$$

where $p$ is the dipole moment. The wave function $\psi$ is written as a linear combination of the wave function of the unperturbed atomic system $\psi_k$:

$$\psi(t) = \sum_k a_k(t)\psi_k = a_0\psi_0 + a_1\psi_1 \tag{5}$$

which leads to a system of differential equations for the coefficients $a_k(t)$:

$$\frac{da_k}{dt} = i\omega_k a_k + \sum_j \frac{i}{2\hbar}p_{k,j}[\tilde{\mathcal{E}}_1 e^{i\omega_{\ell,1}t} + c.c.]a_j \tag{6}$$

Spelling it out:

$$\begin{aligned}\frac{da_0}{dt} &= 0 + \frac{i}{2\hbar} p_{01} \tilde{\mathcal{E}}_1^* e^{i\omega_{\ell,1}t} a_1 \\ \frac{da_1}{dt} &= i\omega_{\ell,1} a_1 + \frac{i}{2\hbar} p_{01} \tilde{\mathcal{E}}_1 e^{i\omega_{\ell,1}t} a_0\end{aligned} \tag{7}$$

The "rotating frame" approximation for this particular situation is:

$$\begin{aligned}a_0 &= c_0 \\ a_1 &= e^{-i\omega_{\ell,1}t}\ c_1\end{aligned} \tag{8}$$

Substituting:

$$\begin{aligned}\frac{dc_0}{dt} &= \frac{i}{2\hbar} p_{1,0} \tilde{\mathcal{E}}_1^*(t)\ c_1 \\ \frac{dc_1}{dt} &= -i\Delta_1 c_1 + \frac{i}{2\hbar} p_{0,1} \tilde{\mathcal{E}}_1(t) c_0\end{aligned} \tag{9}$$

This systems takes a simpler form is we define the Rabi frequency as:

$$\tilde{E}_1 = \frac{i}{\hbar} p_{1,0} \tilde{\mathcal{E}}_1 \tag{10}$$

Substituting:

$$\boxed{\begin{aligned}\frac{dc_0}{dt} &= 0 + \frac{1}{2}\tilde{E}_1 c_1 + 0 \\ \frac{dc_1}{dt} &= -\frac{1}{2}\tilde{E}_1^* c_0 - i\Delta_1 c_1\end{aligned} \qquad (11)}$$

or in matrix form:

$$\boxed{\frac{d}{dt}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\tilde{E}_1 \\ -\frac{1}{2}\tilde{E}_1^* & i\Delta_1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}} \tag{12}$$

The system of equations (12) is generally easy to solve numerically. One is generally not interested in expressing the results as a matrix of $c$ coefficients, but instead the $2 \times 2$ matrix of the density matrix elements $\rho_{ij} = c_i c_j^*$. The diagonal elements $c_i c_i^*$ represent the populations of the level $i$. The off-diagonal elements $c_i c_j^*$ are a measure of the amplitude excitation at the frequency $\omega_j - \omega_i$, and is directly connected to the polarization.

$$\dot{\rho}_{01} = \dot{c}_0 c_1^* + c_0 \dot{c}_1^* = \frac{1}{2} E (c_1 c_1^* - c_0 c_0^*) + i\Delta c_0 c_1^* \tag{13}$$

and

$$\dot{\rho}_{11} = -\frac{1}{2} c_0 c_1^* - \frac{1}{2} c_0 c_1^* \tag{14}$$

Defining:

$$W = \frac{1}{2}(\rho_{11} - \rho_{00})$$

leads to Bloch's equations:

$$\begin{aligned} \dot{\rho}_{01} &= i\Delta\rho_{01} + EW \\ \dot{W} &= -\mathrm{Re} E \rho_{01}^* \end{aligned} \tag{15}$$

$$\dot{\rho}_{11} - \dot{\rho}_{00} = \frac{2p}{\hbar}\left[ i\rho_{01}\tilde{E}^* - i\rho_{10}\tilde{E} \right] \tag{16}$$

$$\dot{\rho}_{01} = i\omega_0 \rho_{01} + \frac{ip\tilde{E}}{\hbar}\left[\rho_{11} - \rho_{00}\right] \tag{17}$$

We define u, v, w as:

$$\begin{aligned} \rho_{01} &= (u + iv) e^{i(\omega t + \varphi)} \\ w &= \rho_{11} - \rho_{00} \end{aligned} \tag{18}$$

to get the Maxwell-Bloch equations:

$$\boxed{\begin{aligned} \dot{u} &= (\omega_0 - \omega_\ell - \dot{\varphi})v - \frac{u}{T_2} && (19) \\ \dot{v} &= -(\omega_0 - \omega_\ell - \dot{\varphi})u - \kappa\mathcal{E}w - \frac{v}{T_2} && (20) \\ \dot{w} &= \kappa\mathcal{E}v - \frac{w - w_0}{T_1} && (21) \end{aligned}}$$

The quantity $\kappa\mathcal{E}$ with $\kappa = p/\hbar$ is the Rabi frequency. $T_1$ and $T_2$ are respectively the energy and phase relaxation times. Most of the energy conserving relaxations are generally lumped in the phase relaxation time $T_2$

Defining:

$$\begin{aligned} \tilde{Q} &= -ic_0c_1^* \\ W &= c_1c_1^* - c_0c_0^* \end{aligned} \tag{22}$$

leads to:

$$\boxed{\begin{aligned} \dot{\tilde{Q}} &= i(\omega_0 - \omega_\ell)\tilde{Q} - \kappa\tilde{\mathcal{E}}w - \frac{\tilde{Q}}{T_2} && (23) \\ \dot{w} &= \frac{\kappa}{2}[\tilde{Q}^*\tilde{\mathcal{E}} + \tilde{Q}\tilde{\mathcal{E}}^*] - \frac{w - w_0}{T_1} && (24) \\ \frac{\partial\tilde{\mathcal{E}}}{\partial z} &= -\frac{\mu_0\omega_\ell c}{2n}\int_0^\infty \tilde{Q}(\omega_0')g_{inh}(\omega_0' - \omega_{ih})d\omega_0'. && (25) \end{aligned}}$$

.

# References