$2\ \pi$ pulse soliton

1. Self-induced transparency

Self-induced transparency is one the first examples of soliton propagation in optics, based on coherent resonant propagation in an absorber. It applies mostly to inhomogeneously broadened systems, for which the (undamped) Bloch's interaction equations apply. The purpose of this exercise is to derive the form of the steady state propagating pulse, the $2\ \pi$ sech, in the simpler narrow line limit. The extension to the inhomogeneous broadening case adds a lot of mathematics but little of physics. It is essential when considering the evolution of a pulse towards steady state, but not necessary to derive the steady state. We therefore consider the system of Bloch's Maxwell equations in the slowly varying approximation, to be reduced to:

$$\dot{u} = (\omega_0 - \omega - \dot{\varphi})v \tag{1}$$
$$\dot{v} = -(\omega_0 - \omega - \dot{\varphi})u - \kappa\mathcal{E}w \tag{2}$$
$$\dot{w} = \kappa\mathcal{E}v \tag{3}$$

where the initial value for w at $t = -\infty$ is

$$w_0 = -pN_0. \tag{4}$$

p is the dipole moment of the transition, expectation value of the dipole moment operator $e\mathbf{r}$ between ground state and upper state of the two level transition $\langle 0|e\mathbf{r}|1\rangle$ In these equations, the light electric field is:

$$E(z,t) = \frac{1}{2}\tilde{\mathcal{E}}(z,t)e^{i(\omega t - kz)} = \frac{1}{2}\mathcal{E}(z,t)e^{i[\omega t + \varphi(z,t) - kz]}, \tag{5}$$

where $\tilde{\mathcal{E}}$ is the complex envelope, $\mathcal{E}$ the real amplitude, φ the phase, all slowly varying. ω is the average carrier frequency of the light, and ω_0 the transition frequency of the two-level system. The propagation equation, in terms of $\mathcal{E}$ and φ, in a retarded frame of reference (propagating at the group velocity of the host medium):

$$\frac{\partial\mathcal{E}(z,t)}{\partial z} = -\frac{\mu_0\omega c}{2n}v(z,t) \tag{6}$$
$$\frac{\partial\varphi}{\partial z} = -\frac{\mu_0\omega c}{2n}\frac{u(z,t)}{\mathcal{E}(z,t)}, \tag{7}$$

where we have explicitly written the (retarded) time t and space z dependence of the functions [dependence omitted in Eqs. (1), (2), (3) to simplify the notations].

The purpose of the problem is to find a shape preserving solution of the form:

$$\mathcal{E}(t) = \frac{a}{\tau_s}\text{sech}\left(\frac{t}{\tau_s} - bz\right), \tag{8}$$

and determine the parameters a and b.

We will for simplicity assume to be at exact resonance.

How does the resonance condition simplify the solution? Self consistently u and φ should be zero in Eqs. (1) and Eqs. (7), which reduces the system of equations to be solved at Eqs. (2), (3) and (6)

Guides to the self-preserving solution We have defined in class the tipping angle of the pseudo-polarization θ as:

$$\theta(z) = \int_{-\infty}^{\infty} \kappa \mathcal{E} dt \tag{9}$$

This "tipping angle" of the polarization a time integrated quantity, characteristic of the pulse at a particular position. We can define a time dependent quantity, which is the tipping angle of the polarization as the pseudo-polarization Bloch vector evolves under the influence of the electric field. To avoid confusion in notation, we will call this quantity β:

$$\beta = \int_{-\infty}^{t} \kappa \mathcal{E}(t') dt' \tag{10}$$

This function β will play a central role in finding the steady state solution of Bloch-Maxwell's equations. In order to find the steady state condition, all quantities will be expressed as a function of β. Note that in the retarded frame of reference that we have chosen, the 2π pulse is a shape preserving envelope at an envelope velocity V. Therefore, the key condition is:

$$\frac{\partial \mathcal{E}}{\partial z} = -\frac{1}{V}\frac{\partial \mathcal{E}}{\partial t}. \tag{11}$$

By substitution into Bloch's equation, derive the equation:

$$\frac{\partial^2 \beta}{\partial t^2} = \frac{1}{\tau^2}\sin\beta \tag{12}$$

and show that this equation corresponds to the equation of motion of an inverted pendulum.

Hint: Remember the geometric representation of Bloch's equations. Use the fact that the pseudo-polarization vector has a constant length $N_0 p$, and rotates by an angle $\beta(t)$. Express the coordinates v and w in terms of these quantities.

It is obvious from the vector model that the time evolution of the components of pseudo-polarization vector is:

$$\begin{aligned} u &= 0 \\ v &= N_0 p \sin\beta \\ w &= w_0 \cos\beta = -N_0 p \cos\beta \end{aligned} \tag{13}$$

From the propagation equation (6), taking into account Eq. (11):

$$\frac{1}{V}\frac{\partial \kappa \mathcal{E}}{\partial t} = -\frac{\mu_0 \omega c \kappa}{2n} v(z,t) = -\frac{\mu_0 \omega c N_0 p \kappa}{2n}\sin\beta. \tag{14}$$

Equation (14) can be condensed by define a parameter τ (dimension of time):

$$\frac{\partial \kappa \mathcal{E}}{\partial z} = \frac{1}{\tau^2}\sin\beta = \frac{\partial^2 \beta}{\partial t^2} \tag{15}$$

In terms of all these parameters, we can define the velocity V of the shape preserving solution as

$$\frac{1}{V} = \frac{\mu_0 \omega c N_0 p \kappa \tau^2}{2n}. \tag{16}$$

Show that the integration of Eq. (12) leads to:

$$\kappa\mathcal{E} = \frac{2}{\tau}\sin\frac{\beta}{2} \tag{17}$$

where

$$\tau = \sqrt{\frac{2n}{\mu_0 \omega c N_0 p \kappa V}} \tag{18}$$

The simplest approach is to start from the solution (17) and get to (12) by derivation.

Derive a differential equation for the quantity $q = \kappa\mathcal{E}\tau/2$ (follow the same steps as above)

We have:

$$\frac{\partial q}{\partial t} = \frac{\partial \frac{\kappa\mathcal{E}\tau}{2}}{\partial t} = \frac{\sin\beta}{2\tau} = \frac{1}{\tau}\sin\frac{\beta}{2}\cos\frac{\beta}{2} = \frac{q}{\tau}\sqrt{1-q^2} \tag{19}$$

Integrate that equation to find the 2 π sech pulse.

$$\frac{dt}{\tau} = \frac{dq}{1\sqrt{1-q^2}} = d\left(\text{sech}^{-1}q\right). \tag{20}$$

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Integrating:

$$\frac{t}{\tau} - \frac{b}{\tau} = \text{sech}^{-1}q = \text{sech}^{-1}\frac{\kappa\mathcal{E}\tau}{2} \tag{22}$$

Substituting the definition of q, we find indeed

$$\mathcal{E}(z,t) = \frac{2}{\kappa\tau}\text{sech}[\frac{1}{\tau}(t - \frac{z}{V})]. \tag{23}$$

It can be verified that indeed the area of this pulse is 2 π.

Properties of sech $\text{sech}^2 x = 1 - \tanh^2 x$

$\frac{d\text{sech}x}{dx} = -\tanh\text{sech}x$

$\frac{d\text{sech}^{-1}x}{dx} = -\frac{i}{x\sqrt{1-x^2}}$

2. Pi-pulse in an amplifier

In an absorber, there are always energy losses, which is why we have to look for a solution where the population is returned to ground state. In an amplifier, we could look for shape-preserving solutions, such that the energy stored in the amplifier compensates for the losses. The most energy can be extracted from an amplifier by a π pulse, since all the inverted atoms are returned to ground state by such a pulse. One has however to introduce some losses, if we want to prevent the pulse energy to grow to infinity with propagation. The simplest form of losses are linear losses introduced by scattering, as in the system of Bloch-Maxwell's equations below.

Consider Bloch's equations for an homogeneously broadened amplifier:

$$\dot{u} = (\omega_0 - \omega - \dot{\varphi})v - \frac{u}{T_2} \tag{24}$$

$$\dot{v} = -(\omega_0 - \omega - \dot{\varphi})u - \kappa\mathcal{E}w - \frac{v}{T_2} \tag{25}$$

$$\dot{w} = \kappa\mathcal{E}v, \tag{26}$$

where we have assumed the pulse to be much shorter than the energy relaxation time T_1. For the propagation equations, in terms of $\mathcal{E}$ and φ, written again in a retarded frame of reference (propagating at the group velocity of the host medium), but has now a scattering loss included:

The propagation equation,

$$\frac{\partial\mathcal{E}(z,t)}{\partial z} = -\frac{\mu_0\omega c}{2n}v(z,t) - \frac{\sigma}{2}\mathcal{E}(z,t) \tag{27}$$

$$\frac{\partial\varphi}{\partial z} = -\frac{\mu_0\omega c}{2n}\frac{u(z,t)}{\mathcal{E}(z,t)}, \tag{28}$$

The initial condition for the polarization is $u = 0$, $v = 0$, and

$$w_0 = pN_0, \tag{29}$$

hence positive, in contrast to the case of the absorber treated in the previous problem.

Simplify the problem for the condition of exact resonance

Explain why, in a gain medium at resonance, in the absence of dispersion ($u = 0$), you cannot expect a "slow" propagating solution as in the case of self-induced transparency Because of that condition, in the retarded frame, the self preserving solution will correspond to $\partial\mathcal{E}/\partial z = 0$.

By substitution of $\mathcal{E} = a\text{sech}(t/\tau)$ into Maxwell's Bloch equations, find the shape preserving π pulse

Can an off-resonance/chirped solution exist?

In looking for an off-resonance shape preserving solution, one should consider that a very small velocity correction is of physical importance. We are looking for a distortionless propagation over infinite distance. For an off-resonant pulse, the resonant dispersion of the two-level system is expected to have an effect on the steady state pulse velocity. In that respect, the slowly varying approximation that led to Eq. (28) is not correct in the context of infinite distortionless propagation. This equation results from writing:

$$\mathcal{E}\left(k^2 - \frac{n^2}{c^2}\omega^2\right) - 2\mathcal{E}\left(\frac{\partial\varphi}{\partial z} - \frac{n^2}{c^2}\frac{\partial\varphi}{\partial t}\right) + \left[\left(\frac{\partial\varphi}{\partial z}\right) - \frac{n^2}{c^2}\dot{\varphi}^2\right]\mathcal{E} = -\mu 0\omega^2 u. \tag{30}$$

where the first term of this equation is put equal to zero, and the third one is neglected. This is a un-necessary approximation. The correct equation is:

$$\mathcal{E}\left[\left(k - \frac{\partial\varphi}{\partial z}\right)^2 - \frac{n^2}{c^2}\left(\omega + \dot{\varphi}\right)^2\right] = \mu_0(\omega + \dot{\varphi}^2 u \tag{31}$$

The phase velocity being:

$$V_{ph} = \frac{\omega + \dot{\varphi}}{k - \frac{\partial\varphi}{\partial z}} \tag{32}$$

The second Bloch equation can be re-written in terms of V_{ph}

For the final derivation, assume $\omega = \omega_0 + 1/T_2$.

Look for a sech solution propagating at the velocity V_{ph}