Laser Physics (Physics 464)
Homework I, due Wednesday, September 6, 2023

Transverse Doppler shift

Year 202?, the Artemis NASA mission is on the moon with a high resolution wavemeter. Jan Hall's one micron wavelength laser with one Herz bandwidth is beamed at that detector. Is that sufficiently narrow bandwidth? If not, what is the minimum bandwidth required? How long do you have to track the detector on the moon with your laser beam in order to do this measurement? Assume a circular orbit for the moon around the earth.

Answer

Radius of the moon orbit: about 1 s at the speed of light ≈ 300,000 km or $3 \cdot 10^8$ m (a good approximation of $3.48 \cdot 10^8$ m).
The angular velocity of the moon is $\Omega_R = 2\pi/(27.3 \text{ day})$, or in s^{-1}

$$\frac{2\pi}{27.3 \times 24 \times 60 \times 60} = 2.66 \cdot 10^{-6}\text{s}^{-1}$$

. The transverse velocity (velocity of the moon) is:

$$v = R\Omega = 3.48 \cdot 10^8 \times 2.66 \cdot 10^{-6} \approx 900\text{m/s}.$$

The transverse Doppler shifted frequency is thus:

$$\nu_d = \nu_0 \sqrt{1 - \frac{v^2}{c^2}} = \nu_0 \left[\sqrt{1 - \frac{8 \cdot 10^5}{9 \cdot 10^{16}}} \right]$$

Try punching that on your calculator! It is simpler on the back of an envelope.
The Doppler shift is:

$$\Delta\nu = \nu_d - \nu_0 = \nu_0 \times \left\{ 1 - \sqrt{1 - 9 \cdot 10^{-12}} \right\} \approx \nu_0 \approx \nu_0 \left[1 - (1 - 4.5 \cdot 10^{-12}) \right]$$

At the wavelength of 1 μm, $\nu_0 = 3 \cdot 10^{14}$ s^{-1}, and $\Delta\nu_d = 3 \cdot 10^{14} \times 4.5 \cdot 10^{-12} =$ 1.35 kHz Conclusion: easily measurable. Measurement time > 1 ms.

Longitudinal Doppler shift

The orbit of the moon is not circular, but has a fairly large eccentricity of 13%. The moon-earth distance varies periodically. Take that variation to be sinusoidal. What is the Doppler shift at the point(s) where the longitudinal component of the velocity is maximum?

Answer - the suggested (but unreal) way

Suppose the laser source is floating in space at the center of the ellipse (not realistic indeed). You approximate the distanced as:

$$d = R\,(1 \pm 0.065\cos 2\Omega t)$$

The longitudinal velocity is $dd/dt = 0.13R\Omega \sin 2\Omega t$ with the maximum value of $0.13R\Omega$.

The Doppler shift is $\nu_0 \times 0.13R\Omega/c = 3 \cdot 10^{14} \times 0.13 \times 3 \cdot 10^8 \times 2.66 \cdot 10^{-6}/(3 \cdot 10^8) \approx 800$ MHz.

Answer - the earth is at the focus of the ellipse

If e is the eccentricity, s the shortest distance, the longest distance is $e + s$ The amplitude of the distance oscillation is $e = 0.13R$. The longitudinal velocity is $dd/dt = 0.13R\Omega \sin \Omega t$ with the maximum value of $0.13R\Omega$.

Interesting that the wrong approach gives the right Doppler shift. Note the frequency (of the periodicity) doubling with the source in the center rather than at the focus of the ellipse.

Spacecraft

In the 70's at the peak of excitement about space exploration, a propulsion scheme was proposed to visit another galaxy. Since it is extremely costly to bring up fuel to a spacecraft, it was proposed to attach huge reflectors to the spacecraft, and to "push" it with a powerful light beam. Another approach is to "beam up" energy with a powerful laser, which is then collected by solar panels, and converted into energy to power an ionic engine to propel the satellite or spacecraft.

1. To get a comparison between the two approaches, assume that the energy conversion is 10% efficient (from laser light to propulsion). Assume a continuous laser beam of 1 MW power, being completely collected by solar panels of 100 m diameter, and applied for 200 seconds.

Assuming a mass of 1000 kg, what will be the velocity increase of the spacecraft after 200 s or irradiation?

2. For comparison, replace the solar panels by perfectly reflecting mirrors. The laser beam is now used to push the spacecraft by radiation pressure. What will be the velocity after 200 seconds (still a 1 MW laser beam)?

Note that the authors of the article had also an answer to the question: "how do you stop the spacecraft". Simple, answered the authors: the space travelers have to attempt communication with a more advanced civilization, screaming "help" in all possible language and form as they are pushed into space!

Solution

10% of the power for 200 seconds corresponds to an energy applied to acceleration of 20 MJ $= mv^2/2$. The velocity is thus:

$$v = \sqrt{\frac{2 \times 20 \cdot 10^6}{1000}} = 200\mathrm{m/s}. \tag{1}$$

In the case of the reflecting mirror, the total number of photons that have impinged on the mirror in 200 s is $200\mathrm{MJ}/\hbar\omega = N$. The recoil momentum per photon is $(\hbar\omega/c)$. In the case of a mirror, there is a double recoil (the photon that got onto the mirror gives on recoil, another recoil as the mirror spits the photon back out). The change in momentum mv (m is the mass, v the velocity, and we start from a zero initial velocity) is equal to the change of momenta due to the photons.

$$mv = 2N\frac{\hbar\omega}{c} = \frac{200\mathrm{MJ}}{\hbar\omega}\frac{2\hbar\omega}{c} = \frac{4 \cdot 10^8}{c} = \frac{4}{3}. \tag{2}$$

The velocity is thus:

$$v = \frac{4}{3}10^{-3}\mathrm{m/s} \tag{3}$$