

Main Fourier Transform Properties

Definitions

Let us consider only the temporal dependence of the electric field.

Real field $E(t)$ The Fourier transform is:

$$\tilde{E}(\Omega) = \mathcal{F}\{E(t)\} = \int_{-\infty}^{\infty} E(t)e^{-i\Omega t} dt = |\tilde{E}(\Omega)|e^{i\Phi(\Omega)} \quad (1)$$

In the definition (1), $|\tilde{E}(\Omega)|$ denotes the spectral amplitude and $\Phi(\Omega)$ is the spectral phase. Complex quantities related to the field are typically written with a tilde.

Property: FT(real) \rightarrow symmetric

Since $E(t)$ is a real function, $\tilde{E}(\Omega) = \tilde{E}^*(-\Omega)$ holds.

Proof:

$$\tilde{E}^*(-\Omega) = \int_{-\infty}^{\infty} E^*(t)e^{-[-i(-\Omega t)]} dt = \int_{-\infty}^{\infty} E(t)e^{[-i(\Omega t)]} dt \quad (2)$$

since $E = E^*$ if E is real.

Inverse FT

Given $\tilde{E}(\Omega)$, the time dependent electric field is obtained through the inverse Fourier transform (\mathcal{F}^{-1}):

$$E(t) = \mathcal{F}^{-1}\{\tilde{E}(\Omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(\Omega)e^{i\Omega t} d\Omega \quad (3)$$

Complex field

It is obtained by taking the inverse Fourier transform of only the positive frequencies:

$$\frac{1}{2\pi} \int_0^{\infty} \tilde{E}(\Omega)e^{i\Omega t} d\Omega = \tilde{E}(t) \quad (4)$$

Decomposing in amplitude and phase

$$\tilde{E}(t) = \frac{1}{2} \tilde{\mathcal{E}}(t)e^{i\omega t}.$$

The real field is $\tilde{E}(t) + \text{complex conjugate}$.

Properties

Properties of Fourier transforms

Linear superposition

Shift	\longleftrightarrow	Linear phase
Real	\longleftrightarrow	$E(\Omega) = E^*(-\Omega)$
Product	\longleftrightarrow	Convolution
Derivative	\longrightarrow	$\times i\Omega$
$\times (-it)$	\longleftarrow	Derivative

Specific functions: Square pulse
 Gaussian
 Single sided exponential

Figure 1:

Linear superposition

$$\mathcal{F}[E_1(t) + E_2(t) + E_3(t) + \dots] = \tilde{E}_1(\Omega) + \tilde{E}_2(\Omega) + \tilde{E}_3(\Omega) + \dots$$

-Obvious from the definition.

Shift

$$\mathcal{F}[E(t - \tau)] = \int_{-\infty}^{\infty} E(t - \tau) e^{-i\Omega(t - \tau)} dt = e^{i\Omega\tau} \int_{-\infty}^{\infty} E(t - \tau) e^{-i\Omega t} dt = e^{i\Omega\tau} \tilde{E}(\Omega). \quad (5)$$

and

$$\mathcal{F}[E(t) e^{iat}] = \int_{-\infty}^{\infty} E(t) e^{-i(\Omega - a)t} dt = \tilde{E}(\Omega - a). \quad (6)$$

Derivative

Integration by parts: $\int f dg = [fg]_{-\infty}^{\infty} - \int gdf = -\int gdf$ when G does not extend to infinity. Thus:

$$\mathcal{F} \left[\frac{dE}{dt} \right] = \int_{-\infty}^{\infty} \frac{dE}{dt} e^{-i\Omega t} dt = - \int_{-\infty}^{\infty} E(-i\Omega) e^{-i\Omega t} dt = i\omega \mathcal{F} [E]. \quad (7)$$

For the second derivative, you apply the rule twice:

$$\mathcal{F} \left[\frac{d^2 E}{dt^2} \right] = -\omega^2 \mathcal{F} [E]. \quad (8)$$

Convolution theorem

The convolution of the two functions $\tilde{E}(t)$ and $\tilde{F}(t)$ is:

$$\begin{aligned} \tilde{E}(t) * \tilde{F}(t) &= \int \tilde{E}(t') \tilde{F}(t-t') dt \\ &= \int \tilde{E}(t') \left[\frac{1}{2\pi} \int \tilde{F}(\Omega) e^{\Omega(t-t')} d\Omega \right] dt' \\ &= \frac{1}{2\pi} \int \tilde{F}(\Omega) \left[\int \tilde{E}(t') e^{-i\Omega t'} dt' \right] e^{i\Omega t} d\Omega \\ &= \frac{1}{2\pi} \int \tilde{F}(\Omega) \tilde{E}(\Omega) e^{i\Omega t} d\Omega, \end{aligned} \quad (9)$$

which is the inverse Fourier of the product of the Fourier transforms.

Correlation theorem

The Correlation of the two functions $\tilde{E}(t)$ and $\tilde{F}(t)$ is:

$$\begin{aligned} \tilde{E}(t) \otimes \tilde{F}(t) &= \int \tilde{E}^*(t') \tilde{F}(t'-t) dt \\ &= \int \tilde{E}^*(t') \left[\frac{1}{2\pi} \int \tilde{F}(\Omega) e^{\Omega(t'-t)} d\Omega \right] dt' \\ &= \frac{1}{2\pi} \int \tilde{F}(\Omega) \left[\int \tilde{E}^*(t') e^{-i\Omega t'} dt' \right] e^{i\Omega t} d\Omega \\ &= \frac{1}{2\pi} \int \tilde{F}(\Omega) \tilde{E}^*(\Omega) e^{i\Omega t} d\Omega, \end{aligned} \quad (10)$$

which is the inverse Fourier of the product of one Fourier transform by the complex conjugate of the other.

Autocorrelation

The Fourier transform of $\tilde{E}(t)$:

$\tilde{E}(t) \otimes \tilde{E}(t)$ is $\tilde{E}^*(\Omega)\tilde{E}(\Omega)$ or $|\tilde{E}(\Omega)|^2$ which is real. It is also proportional to the spectral intensity corresponding to $\tilde{E}(t)$. Since the Fourier transform is real, the autocorrelation $\tilde{E}(t) \otimes \tilde{E}(t)$ must be symmetric.

The autocorrelation is constructed by making the overlap integral of the function $E(t)$ with itself delayed by τ : $E(t - \tau)$. It should be clear that positive or negative τ will give the same result.

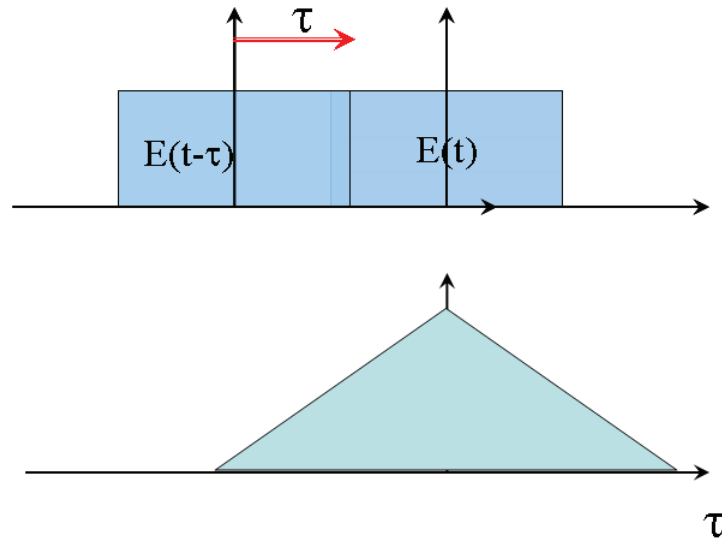


Figure 2: Autocorrelation of a rectangle.

Fourier transform of a square

Take the function $E(t)$ to be 0 from $t = -\infty$ to $t = -a/2$, = 1 from $t = -a/2$ to $t = a/2$, 0 from $t = a/2$ to $t = \infty$.

$$\mathcal{F}[E] = \int_{-a/2}^{a/2} e^{-i\Omega t} dt = i \left[\frac{e^{-i\Omega a/2}}{\Omega} - \frac{e^{i\Omega a/2}}{\Omega} \right] = \frac{a \sin \Omega a/2}{\Omega/2} = (a/2) \text{sinc}(\Omega a/2). \quad (11)$$