Main Fourier Transform Properties

Definitions

Let us consider only the temporal dependence of the electric field.
Real field $E(t)$ The Fourier transform is:

$$\tilde{E}(\Omega) = \mathcal{F}\left\{E(t)\right\} = \int_{-\infty}^{\infty} E(t)e^{-i\Omega t}dt = |\tilde{E}(\Omega)|e^{i\Phi(\Omega)} \tag{1}$$

In the definition (1), $|\tilde{E}(\Omega)|$ denotes the spectral amplitude and $\Phi(\Omega)$ is the spectral phase. Complex quantities related to the field are typically written with a tilde.

Property: FT(real) $\rightarrow$ symmetric

Since $E(t)$ is a real function, $\tilde{E}(\Omega) = \tilde{E}^*(-\Omega)$ holds.
Proof:

$$\tilde{E}^*(-\Omega) = \int_{-\infty}^{\infty} E^*(t)e^{-[-i(-\Omega t)]}dt = \int_{-\infty}^{\infty} E(t)e^{[-i(\Omega t)]}dt \tag{2}$$

since $E = E^*$ if E is real.

Inverse FT

Given $\tilde{E}(\Omega)$, the time dependent electric field is obtained through the inverse Fourier transform ($\mathcal{F}^{-1}$):

$$E(t) = \mathcal{F}^{-1}\left\{\tilde{E}(\Omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\Omega)e^{i\Omega t}d\Omega \tag{3}$$

Complex field

It is obtained by taking the inverse Fourier transform of only the positive frequencies:

$$\frac{1}{2\pi}\int_{0}^{\infty} \tilde{E}(\Omega)e^{i\Omega t}d\Omega = \tilde{E}(t) \tag{4}$$

Decomposing in amplitude and phase

$$\tilde{E}(t) = \frac{1}{2}\tilde{\mathcal{E}}(t)e^{i\omega t}.$$

The real field is $\tilde{E}(t)$ + complex conjugate.

Properties

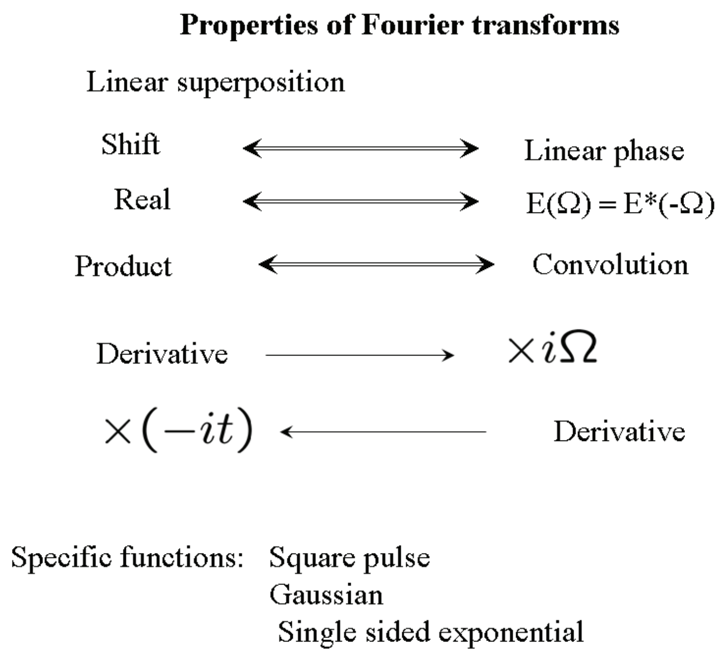

Figure 1:

Linear superposition

$$\mathcal{F}\left[E_1(t) + E_2 1(t) + E_3(t) + \cdot\right] = \tilde{E}_1(\Omega) + \tilde{E}_2(\Omega) + \tilde{E}_3(\Omega) + \cdot)$$

-Obvious from the definition.

Shift

$$\mathcal{F}\left[E(t-\tau)\right] = \int_{-\infty}^{\infty} E(t-\tau)e^{-i\Omega(t-\tau)}dt = e^{i\Omega\tau}\int_{-\infty}^{\infty} E(t-\tau)e^{-i\Omega t}dt = e^{i\Omega\tau}\tilde{E}(\Omega). \tag{5}$$

and

$$\mathcal{F}\left[E(t)e^{iat}\right] = \int_{-\infty}^{\infty} E(t)e^{-i(\Omega-a)t}dt = \tilde{E}(\Omega - a). \tag{6}$$

Derivative

Integration by parts: $\int f dg = [fg]_{-\infty}^{\infty} - \int g df = -\int g df$ when G does not extend to infinity. Thus:

$$\mathcal{F}\left[\frac{dE}{dt}\right] = \int_{-\infty}^{\infty} \frac{dE}{dt} e^{-i\Omega t} dt = -\int_{-\infty}^{\infty} E(-i\Omega) e^{-i\Omega t} dt = i\omega \mathcal{F}\left[E\right]. \tag{7}$$

For the second derivative, you apply the rule twice:

$$\mathcal{F}\left[\frac{d^2E}{dt^2}\right] = -\omega^2 \mathcal{F}\left[E\right]. \tag{8}$$

Convolution theorem

The convolution of the two functions $\tilde{E}(t)$ and $\tilde{F}(t)$ is:

$$\begin{aligned}
\tilde{E}(t) * \tilde{F}(t) &= \int \tilde{E}(t')\tilde{F}(t-t')dt \\
&= \int \tilde{E}(t') \left[\frac{1}{2\pi}\int \tilde{F}(\Omega) e^{\Omega(t-t')} d\Omega\right] dt' \\
&= \frac{1}{2\pi}\int \tilde{F}(\Omega) \left[\int \tilde{E}(t') e^{-i\Omega t'} dt'\right] e^{i\Omega t} d\Omega \\
&= \frac{1}{2\pi}\int \tilde{F}(\Omega)\tilde{E}(\Omega) e^{i\Omega t} d\Omega,
\end{aligned} \tag{9}$$

which is the inverse Fourier of the product of the Fourier transforms.

Correlation theorem

The Correlation of the two functions $\tilde{E}(t)$ and $\tilde{F}(t)$ is:

$$\begin{aligned}
\tilde{E}(t) \otimes \tilde{F}(t) &= \int \tilde{\mathcal{E}}^*(t')\tilde{F}(t'-t)dt \\
&= \int \tilde{E}^*(t') \left[\frac{1}{2\pi}\int \tilde{F}(\Omega) e^{\Omega(t'-t)} d\Omega\right] dt' \\
&= \frac{1}{2\pi}\int \tilde{F}(\Omega) \left[\int \tilde{E}^*(t') e^{-i\Omega t'} dt'\right] e^{i\Omega t} d\Omega \\
&= \frac{1}{2\pi}\int \tilde{F}(\Omega)\tilde{E}^*(\Omega) e^{i\Omega t} d\Omega,
\end{aligned} \tag{10}$$

which is the inverse Fourier of the product of one Fourier transform by the complex conjugate of the other.

Autocorrelation

The Fourier transform of $\tilde{E}(t)$:
$\tilde{E}(t) \otimes \tilde{E}(t)$ is $\tilde{E}^*(\Omega)\tilde{E}(\Omega)$ or $|\tilde{E}(\Omega)|^2$ which is real. It is also the proportional to the spectral intensity corresponding to $\tilde{E}(t)$. Since the Fourier transform is real, the autocorrelation $\tilde{E}(t) \otimes \tilde{E}(t)$ must be symmetric.

The autocorrelation is constructed by making the overlap integral of the function $E(t)$ with itself delayed by τ: $E(t-\tau)$. It should be clear htat positive or negative τ will give the same result.

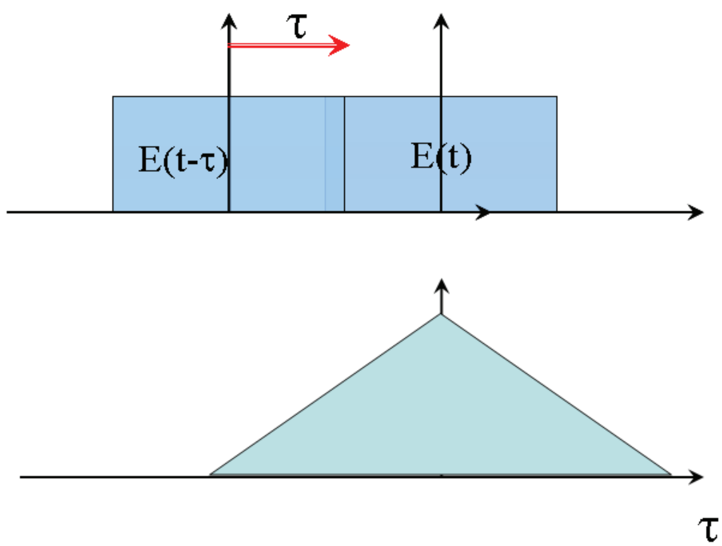

Figure 2: Autocorrelation of a rectangle.

Fourier transform of a square

Take the function $E(t)$ to be 0 from $t = -\infty$ to $t = -a/2$, = 1 from $t = -a/2$ to $t = a/2$, 0 from $t = a/2$ to $t = -\infty$.

$$\mathcal{F}[E] = \int_{-a/2}^{a/2} e^{-i\Omega t} dt = i\left[\frac{e^{-i\Omega a/2}}{\Omega} - \frac{e^{i\Omega a/2}}{\Omega}\right] = \frac{a}{2}\frac{\sin \Omega a/2}{\Omega/2} = (a/2)\text{sinc}(\Omega a/2). \tag{11}$$