Laser Physics I (Physics 464)
Homework III, due Monday, October 19, 2026

1 Evolution of the pulse energy

Find an evolution equation for the pulse energy
$W = \frac{1}{2}\epsilon_0 cn \int_{-\infty}^{\infty} \mathcal{E}^2 dt = \frac{1}{2}\sqrt{\epsilon/\mu_0} \int_{-\infty}^{\infty} \mathcal{E}^2 dt$.
integrate the field envelope propagation over time, after multiplying both sides by $\mathcal{E}$ and using the third Bloch equation.
Why does it make physical sense?

2 Find an evolution equation for the pulse average frequency

$\dot{\varphi}\rangle$ is the average phase derivative (average deviation of the frequency from ω_ℓ):

$$\langle\dot{\varphi}\rangle = \frac{\int_{-\infty}^{\infty} \mathcal{E}^2 \dot{\varphi} dt}{\int_{-\infty}^{\infty} \mathcal{E}^2 dt} = \frac{1}{2W}\left[\sqrt{\frac{\epsilon}{\mu_0}} \int_{-\infty}^{\infty} \mathcal{E}^2 \dot{\varphi} dt\right]$$

Multiply both sides by W and take the derivative

$$\begin{aligned} \frac{d(W\langle\dot{\varphi}\rangle)}{dz} &= W\frac{d\langle\dot{\varphi}\rangle}{dz} + \langle\dot{\varphi}\rangle\frac{dW}{dz} \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu_0}}\frac{d}{dz}\int_{-\infty}^{\infty} \mathcal{E}^2 \dot{\varphi} dt \end{aligned}$$

The last term can be calculated inserting the time derivative of the phase propagation equation in this equation.

3 SOLUTIONS

3.1 Evolution equation for the pulse energy

Find an evolution equation for the pulse energy
$W = \frac{1}{2}\epsilon_0 cn \int_{-\infty}^{\infty} \mathcal{E}^2 dt = \frac{1}{2}\sqrt{\epsilon/\mu_0} \int_{-\infty}^{\infty} \mathcal{E}^2 dt$.
integrate the field envelope propagation over time, after multiplying both sides by $\mathcal{E}$ and using the third Bloch equation.

$$\begin{aligned}
\frac{dW}{dz} &= \sqrt{\frac{\epsilon}{\mu_0}} \int_{-\infty}^{\infty} \mathcal{E} \frac{\partial \mathcal{E}}{\partial z} dt \\
&= -\frac{\mu_0 \omega_\ell \; c}{2} \sqrt{\frac{\epsilon}{\mu_0}} \int_{-\infty}^{\infty} \int_{0}^{\infty} v(\omega_0') \mathcal{E} g_{inh}(\omega_0' - \omega_{ih}) d\omega_0' dt \\
&= -\frac{\hbar \omega_\ell}{2p} \int_0^{\infty} \left[w_\infty(\omega_0') - w_0(\omega_0') \right] g_{inh}(\omega_0' - \omega_{ih}) d\omega_0' \qquad (1)
\end{aligned}$$

The population difference (per unit volume) $(w_\infty - w_0)/p$ integrated over the inhomogeneous transition is a measure of the energy stored in the medium, as a consequence of the energy lost by the pulse, dW/dz.

3.2 Evolution equation for the pulse average frequency

$\langle \dot{\varphi} \rangle$ is the average phase derivative (average deviation of the frequency from ω_ℓ):

$$\langle \dot{\varphi} \rangle = \frac{\int_{-\infty}^{\infty} \mathcal{E}^2 \dot{\varphi} dt}{\int_{-\infty}^{\infty} \mathcal{E}^2 dt} = \frac{1}{2W} \left[\sqrt{\frac{\epsilon}{\mu_0}} \int_{-\infty}^{\infty} \mathcal{E}^2 \dot{\varphi} dt \right]$$

Multiply both sides by W and take the derivative

$$\begin{aligned}
\frac{d(W\langle \dot{\varphi} \rangle)}{dz} &= W \frac{d\langle \dot{\varphi} \rangle}{dz} + \langle \dot{\varphi} \rangle \frac{dW}{dz} \\
&= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu_0}} \frac{d}{dz} \int_{-\infty}^{\infty} \mathcal{E}^2 \dot{\varphi} dt
\end{aligned}$$

The last term of this equation is:

$$\frac{1}{2} \sqrt{\frac{\epsilon}{\mu_0}} \int_{-\infty}^{\infty} \left(\mathcal{E}^2 \frac{\partial \dot{\varphi}}{\partial z} + \dot{\varphi} \frac{\partial \mathcal{E}^2}{\partial z} \right) dt \qquad (2)$$

Remembering Bloch's equation:

$$\dot{u} = \Delta\omega v \tag{3}$$
$$\dot{v} = -\Delta\omega u - \kappa\mathcal{E} w \tag{4}$$
$$\dot{w} = \kappa\mathcal{E} v \tag{5}$$

and the propagation equations:

$$\frac{\partial\mathcal{E}}{\partial z} = -\frac{\mu_0\omega_\ell c}{2n}\int_0^\infty v(\omega_0')g_{inh}(\omega_0'-\omega_{ih})d\omega_0' \tag{6}$$
$$\frac{\partial\varphi}{\partial z} = -\frac{\mu_0\omega_\ell c}{2n}\int_0^\infty \frac{u(\omega_0')}{\mathcal{E}}g_{inh}(\omega_0'-\omega_{ih})d\omega_0'. \tag{7}$$

Inserting the time derivative of the phase propagation equation (7) in Eq. 2:

$$\begin{aligned}&\frac{1}{2}\sqrt{\frac{\epsilon}{\mu_0}}\int_{-\infty}^{\infty}\left(\mathcal{E}^2\frac{\partial\dot{\varphi}}{\partial z}+\dot{\varphi}\frac{\partial\mathcal{E}^2}{\partial z}\right)dt\\ =\ &-\frac{\omega_\ell}{4}\int_{-\infty}^{\infty}dt\int_0^\infty d\omega_0'\ g_{inh}(\omega_0'-\omega_{ih})\left[\dot{u}\mathcal{E}+u\dot{\mathcal{E}}-2v\mathcal{E}\dot{\varphi}\right]\end{aligned}$$

Integrating by parts:

$$\begin{aligned}&= -\frac{\omega_\ell}{2}\int_0^\infty d\omega_0'\int_{-\infty}^{\infty}dt\ g_{inh}(\omega_0'-\omega_{ih})[\dot{u}\mathcal{E}-\dot{\varphi}v\mathcal{E}]\\ &= -\frac{\omega_\ell}{2}\int_0^\infty d\omega_0'\int_{-\infty}^{\infty}dt\ g_{inh}(\omega_0'-\omega_{ih})\left[(\omega_0'-\omega_\ell-\dot{\varphi})v\mathcal{E}\right]\end{aligned} \tag{8}$$

where we have made use of the first Bloch equation in $\dot{u}$. We have already derived an expression for the evolution of the pulse energy density W. Combining Eqs. (1),the definition of the first moment, and (8) yields the following expression for the evolution with propagation distance of the pulse carrier frequency:

$$\boxed{\frac{d\langle\dot{\varphi}\rangle}{dz} = \frac{\omega_\ell}{2\kappa W}\int_0^\infty g_{inh}(\omega_0'-\omega_{ih})\left[\omega_0'-\omega_\ell-\langle\dot{\varphi}\rangle\right](w_\infty-w_0)\,d\omega_0'.} \tag{9}$$

The frequency shift is proportional to the overlap integral of the lineshape $g_{inh}(\omega_0'-\omega_{ih})$ with the (frequency dependent) change of the inversion $(w_\infty - w_0)$ times the detuning $(\omega_0'-\omega_\ell-\langle\dot{\varphi}\rangle)$. The ratio of absorbed energy (which is proportional to $(w_\infty - w_0)$) to the pulse energy W is maximum in the

weak pulse limit ($\theta \ll 1$). Therefore, the frequency pushing as described by Eq. (9) is important in the weak pulse limit, and for narrow lines [$T_2 \rightarrow \infty$; $g_{inh}(\omega_0' - \omega_{ih}) \approx \delta(0)$].

Bloch's equations can be solved analytically in the weak, short pulse limit, i.e., for pulses that do not induce significant changes in population and have a duration short compared to the phase relaxation time T_2. The interaction equation can be written in the integral form:

$$\tilde{Q}(t) = \int_{-\infty}^{t} \kappa \mathcal{E} w e^{-i[(\omega_0' - \omega_\ell)t' - \varphi(t')]} dt' \tag{10}$$

For weak pulses ($w \approx w_0$) and the right hand side of Eq. (10) at $t = \infty$ is proportional to the Fourier transform of $\kappa\tilde{\mathcal{E}}w$. Thus we have:

$$\begin{aligned} |\tilde{Q}|^2 = u^2 + v^2 &= \kappa^2 w_0^2 |\tilde{\mathcal{E}}(\omega_0' - \omega_\ell)|^2 \\ &\approx -2w_0(w_\infty - w_0) \end{aligned} \tag{11}$$

where $\tilde{\mathcal{E}}(\omega_0' - \omega_\ell)$ is the amplitude of the Fourier transform of the field envelope at the line frequency ω_0'. The last equality results from the conservation of the length of the pseudo-polarization vector ($u^2 + v^2 + w^2 = w_0^2 = \text{constant}$). The approximation is made that the change in population is small: $w_\infty^2 = [w_0 + (w_\infty - w_0)]^2 \approx w_0^2 + 2w_0(w_\infty - w_0)$. Insertion of the approximation (11) into the energy evolution equation (1) leads to the expected result that the absorbed energy is proportional to the overlap of the pulse and line spectra:

$$\frac{dW}{dz} = \frac{\kappa\omega_\ell \; w_0}{4} \int_0^\infty |\tilde{\mathcal{E}}(\omega_0' - \omega_\ell)|^2 g_{inh}(\omega_0' - \omega_{ih}) d\omega_0' \tag{12}$$

A similar expression is found by inserting the approximation (11) into the frequency evolution equation (9):

$$\frac{d\langle\dot{\varphi}\rangle}{dz} = \frac{\kappa\omega_\ell w_0}{2W} \int_0^\infty (\omega_0' - \omega_\ell - \langle\dot{\varphi}\rangle) |\tilde{\mathcal{E}}(\omega_0' - \omega_\ell)|^2 g_{inh}(\omega_0' - \omega_{ih}) d\omega_0' \tag{13}$$

Comparing expressions (12) and (13) leads to the conclusion that the frequency shift with distance is largest for sharp lines (compared to the pulse bandwidth), and pulses off resonance by approximately their bandwidth. The maximum possible shift is about half a pulse bandwidth per absorption length.

The frequency shift with distance is a manifestation of the fact that the response time of matter is finite. We have already alluded to this in Chapter 1, when we wrote the polarization as a convolution integral, expressing that the instantaneous polarization is not an immediate function of the electric field of the light, but a function of the history of the field. Such convolution expressions can be obtained directly by Fourier transformation of Bloch's equations.

To appreciate physically why the carrier frequency should not be a conserved quantity, let us consider a square pulse sent through a near resonant absorbing medium at the average carrier frequency ω_ℓ where $\omega_\ell < \omega_{ih}$. High frequency components within the pulse spectrum are closer to the center of the inhomogeneous transition and, thus, are absorbed more than low frequency components. The result is a continuous shift of the pulse spectrum to lower frequencies. Essentially the same phenomenon has been discussed in Chapter 1 (Section 1.2.2). There, the cause was a frequency dependent imaginary part of the dielectric constant. It is left to a problem to find the connection between both descriptions. Another way to describe the frequency pushing is in the time domain. Initially — i.e. prior to the arrival of the light pulse — there were no induced dipoles, since there was no field present to induce them. The initial value of the susceptibility χ (or, using the notations of this chapter, the pseudo-polarization Q or u, v) is thus 0. As the leading edge of the pulse enters the medium, it excites the electronic dipoles, resulting in a change of χ and in a change of the refractive index, respectively. As we have seen before this results in a pulse chirp which can manifest itself in a shift of the average pulse frequency.