Appendix A

Phase Shifts upon Transmission and Reflection

Most often, phase shifts at interfaces are a simple consequence of energy conservation. Conversely, the phase shift properties in simple devices can be used to determine the direction of the flow of energy. A few simple examples are given here.

A.1 The symmetrical interface

Let us consider first the very simple situation sketched in Fig. A.1. The interface can be a mirror with a reflecting coating on the front face and an antireflection coating on the back face. We are only interested in fields propagating *outside* the mirror. The energy conservation relation between the reflected (field reflection coefficient $\tilde{r}$) and transmitted (field transmission coefficient $\tilde{t}$) waves implies:

$$|\tilde{r}|^2 + |\tilde{t}|^2 = 1, \tag{A.1}$$

where we assumed a unity field amplitude incident from the left.

Taking next a field incident from the right, we have:

$$|\tilde{r}'|^2 + |\tilde{t}'|^2 = 1, \tag{A.2}$$

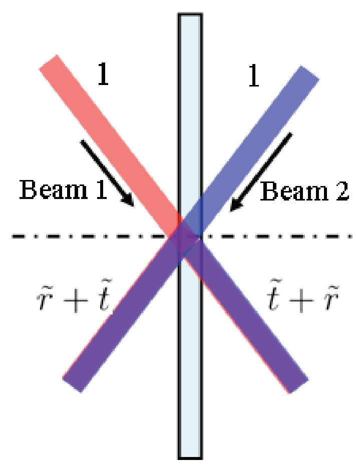

Figure A.1: Reflection and transmission by an interface between two identical media

Another relation can be found by summing the intensities on both sides of the symmetry line (dash-dotted line):

$$|\tilde{r}+\tilde{t'}|^2+|\tilde{r'}+\tilde{t}|^2=2. \tag{A.3}$$

Combination of Eqs. (A.1) and (A.3) leads to

$$2[\tilde{r}\tilde{t'}^*+\tilde{r'}^*\tilde{t}]=0, \tag{A.4}$$

which implies that the phase shifts upon transmission and reflection are complementary in the case of symmetric intefaces:

$$\varphi_r-\varphi_t=\frac{\pi}{2}. \tag{A.5}$$

It is because of the latter phase relation that the antiresonant ring reflects back all the incident radiation, and has zero losses if $|\tilde{r}|^2=|\tilde{t}|^2=0.5$. In the case of zero phase shift in transmission $\varphi_t=0$, Eq. (A.4) implies $r=-r'^*$, which corresponds to conservative coupling.

Note that Eq. (A.5) is not necessarily true if the structure is not symmetric, such as for a Fabry-Perot with different coatings on both sides. In the case, Eq. (A.4) has to be replaced by:

$$[\tilde{r}_1\tilde{t}_2^*+\tilde{r}_1^*\tilde{t}_2+\tilde{r}_2\tilde{t}_1^*+\tilde{r}_2^*\tilde{t}_1]=0, \tag{A.6}$$

where the indices 1 and 2 refer to the reflection/transmittion of beam 1 and 2, respectively. Equation (A.5) has then to be replaced by:

$$\cos(\varphi_{r1}-\varphi_{t2})+\cos(\varphi_{r2}-\varphi_{t1})=0. \tag{A.7}$$

A.2 Coated interface between two different dielectrics

Let us consider – as in Fig. A.2 – a partially reflecting coating at an interface between air (index 1) and a medium of index n. A light beam of amplitude $\mathcal{E}_1=1/\sqrt{\cos\theta_1}$ is incident from the air, at an angle of incidence θ_1. The transmitted beam is refracted at

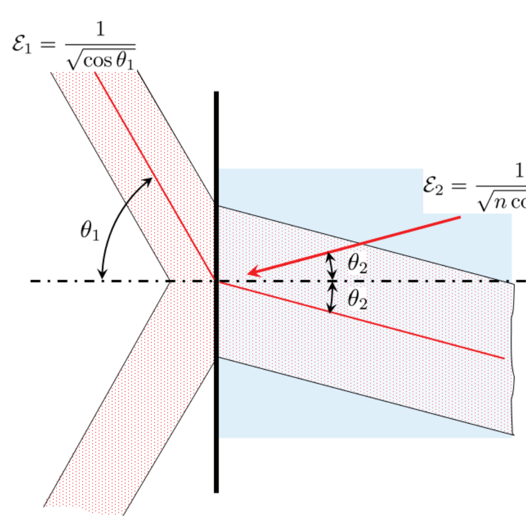

Figure A.2: Reflection and transmission by an interface between air and a dielectric.

the angle θ_2, and has an amplitude $\tilde{t}_1/\sqrt{cos\theta_1}$. The reflected beam has an amplitude $\tilde{r}_1/\sqrt{cos\theta_1}$. We take the vertical (orthogonal to the figure) dimension of the beam to be unity, as well as the distance covered by the beam on the interface in the plane of the figure. To calculate energy conservation, we compare the products $n_i|\tilde{\mathcal{E}}|^2 A$ where $n_i = 1$ left of the interface, $n_i = n$ right of the interface, and $A = 1 \times \cos\theta$. As in the previous section, we will be considering a similar beam incident from the right, with an amplitude $\mathcal{E}_2 = 1/\sqrt{n\cos\theta_2}$ incident at an angle θ_2 on the dielectric/air interface. The choice of these incident electric field amplitudes is such that the same "energy" products $n_i|\tilde{\mathcal{E}}|^2 A = A$ apply on both sides of the interface, above the dash-dotted line in Fig. A.2.

Energy conservation leads to the relation:

$$|\tilde{r}_1|^2 + |\tilde{t}_1|^2 \frac{n\cos\theta_2}{\cos\theta_1} = 1, \tag{A.8}$$

where we took into account the change in beam cross section upon refraction. We have a similar energy conservation equation for a beam of amplitude $\mathcal{E}_2 = 1/\sqrt{n\cos\theta_2}$ incident at an angle θ_2 on the dielectric/air interface:

$$|\tilde{r}_2|^2 + |\tilde{t}_2|^2 \frac{\cos\theta_1}{n\cos\theta_2} = 1. \tag{A.9}$$

From Eqs. (A.8) and (A.9) we get directly the relation:

$$|t_1|^2 \cdot |t_2|^2 = T_1 T_2 = (1 - |r_1|^2)(1 - |r_2|^2) = (1 - R_1)(1 - R_2) \tag{A.10}$$

which is a trivial energy conservation equation. The amplitude of the reflection coefficient is equal on both sides of the interface. Since $|\tilde{r}_1|^2 = |\tilde{r}_2|^2$, Eqs (A.8) and (A.9) lead to:

$$\boxed{|t_1|\sqrt{\frac{n\cos\theta_2}{\cos\theta_1}} = |t_2|\sqrt{\frac{\cos\theta_1}{n\cos\theta_2}}.} \tag{A.11}$$

The amplitudes of the transmission coefficients are not equal, but in the ratio $|t_2|/|t_1| = n\cos\theta_2/\cos\theta_1$, a relation that satisfies Fresnel equations, and results simply from energy conservation.

In order to find a relation between the phase shift upon transmission and reflection, we consider the energy conservation for light incident from the upper half of the figure (the axis of symmetry being the dashed normal to the interface):

$$1 + 1 = \cos\theta_1 \left| \frac{\tilde{r}_1}{\sqrt{\cos\theta_1}} + \frac{\tilde{t}_2}{\sqrt{n\cos\theta_2}} \right|^2 + n\cos\theta_2 \left| \frac{\tilde{r}_2}{\sqrt{n\cos\theta_2}} + \frac{\tilde{t}_1}{\sqrt{\cos\theta_1}} \right|^2 . \tag{A.12}$$

Expanding:

$$2 = |r_1|^2 + |r_2|^2 + |t_2|^2 \frac{\cos\theta_1}{n\cos\theta_2} + |t_1|^2 \frac{n\cos\theta_2}{\cos\theta_2} + (\tilde{r}_1\tilde{t}_2^* + \tilde{r}_1^*\tilde{t}_2)\sqrt{\frac{\cos\theta_1}{n\cos\theta_2}} + (\tilde{r}_2\tilde{t}_1^* + \tilde{r}_2^*\tilde{t}_1)\sqrt{\frac{n\cos\theta_2}{\cos\theta_1}} \tag{A.13}$$

Taking into account the energy conservation relations (A.8) and (A.9), leads to:

$$\boxed{(\tilde{r}_1\tilde{t}_2^* + \tilde{r}_1^*\tilde{t}_2)\cos\theta_1 + (\tilde{r}_2\tilde{t}_1^* + \tilde{r}_2^*\tilde{t}_1)n\cos\theta_2 = 0.} \tag{A.14}$$

We can re-write Eq. (A.14)

$$2|r_1||t_2|\{\cos(\varphi_{r,1} - \varphi_{t,2})\}\cos\theta_1 = -2|r_2||t_1|\{\cos(\varphi_{r,2} - \varphi_{t,1})\}n\cos\theta_2. \tag{A.15}$$

Equation A.14 leads also to the following trigonometric relations between phase shifts upon transmission and reflection:

$$\frac{\cos(\varphi_{r,1} - \varphi_{t,2})}{\cos(\varphi_{r,2} - \varphi_{t,1})} = -1, \tag{A.16}$$

which leads to the relation between phase angles:

$$\varphi_{t,2} - \varphi_{r,1} = \varphi_{r,2} - \varphi_{t,1} + (2n+1)\pi. \tag{A.17}$$

or

$$\varphi_{t,1} + \varphi_{t,2} = \varphi_{r,1} + \varphi_{r,2} + (2n+1)\pi. \tag{A.18}$$

The relations for normal incidence, where $|\tilde{r}_1| = |\tilde{r}_2|$ and $n_1|\tilde{t}_2| = n_2|\tilde{t}_1$, are:

$$n_1(\tilde{r}_1\tilde{t}_2^* + \tilde{r}_1^*\tilde{t}_2) + (\tilde{r}_2\tilde{t}_1^* + \tilde{r}_2^*\tilde{t}_1)n_2 = 0.$$
$$\varphi_{t,1} + \varphi_{t,2} = \varphi_{r,1} + \varphi_{r,2} + (2n+1)\pi.$$

A.3 Matrix method

A.3.1 The "S" matrix

Coatings play an important role in femtosecond optics, by shaping pulses through the amplitude and phase of their reflection coefficient. Matrix methods have been developed to predict all properties of a coating. A coated optical surface can be modeled as a stack of simple thin interfaces separated by propagation in a dielectric. The transmission and reflection of a right propagating field are the result from the interference from all of the layers. To calculate these resultant fields, the right moving and left moving waves must be recorded at each interface. At each simple thin layer interface, the incoming fields (from each direction) are split according to a scattering matrix S defined as:

$$\begin{bmatrix} E_2 \\ E_1' \end{bmatrix} = \begin{bmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2' \end{bmatrix}. \tag{A.19}$$

the electric field subscripts $(1, 2)$ describe whether the field is on the left or right side of the interface, while the no prime and prime discern whether the field is right-propagating or left-propagating, respectively. The coefficients syntax is such that t_{12} and r_{12} describe the transmission and reflection coefficient of the wave starting on side 1 while t_{21} and r_{21} describe the coefficients of a field incoming from side 2.. The S matrix is a 2×2 matrix connecting an input column matrices (incident fields) to an output one (transmitted and reflected fields). The first line of each is a right moving field, and the second one a left moving field. The 2×2 S matrix connects the input fields to the "output" fields

A.3.2 The "M" matrix

The elements of the S matrix have real physical significance; they are the field reflection and transmission amplitudes of a layer. Unfortunately, the S matrix is not useful for building up multilayer surfaces as these matrices cannot be cascaded. What is needed is a matrix defining each layer that can be multiplied by the matrices corresponding to the other layers to create an effective total matrix for the whole structure. Instead of equations defining the relationship between incoming and outgoing fields at each layer, the equations need to define the relationship between the fields on the left and the right side of the layer (regardless of whether they are incoming or outgoing). In other words, in order to cascade the matrix layers, the matrix equation needs to move through the surface spatially (left to right) instead of causally (incoming to outgoing). Such a matrix is defined as the wave-transfer, M, matrix. The column matrices it connects are bound to a surface, with as first line is a right moving field, and as second line a left moving field. The relationship between the M and S matrices is,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{t_{21}} \begin{bmatrix} t_{12}t_{21} - r_{12}r_{21} & r_{21} \\ -r_{12} & 1 \end{bmatrix} \tag{A.20}$$

$$S = \begin{bmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} AD - BC & B \\ -C & 1 \end{bmatrix}. \tag{A.21}$$

A.3.3 Calculating the multilayer transmission and reflection

In order to use the cascaded matrix method, each layer is defined using its physical S matrix. Each S is converted into an M matrix, which are then multiplied together to give an effective total M_T matrix for the entire surface. This ultimate M_T matrix is converted back into a total S_T matrix to extract the effective physical parameters of the multilayer interface. Where the total effective scattering matrix, S_T, has elements that represent,

$$S_T = \begin{bmatrix} \mathcal{T} & \mathcal{R}' \\ \mathcal{R} & \mathcal{T}' \end{bmatrix}, \tag{A.22}$$

where $\mathcal{T}$ and $\mathcal{R}$ are the transmission and reflection coefficients for a beam incoming to the front surface, and $\mathcal{T}'$, $\mathcal{R}'$ are the similar coefficients for a beam incoming to the back surface.

The matrices involved in the calculations of a coating are given in Table A.1.

Type	Free Propagation	Interface
S	$\begin{pmatrix} e^{-inkd} & 0 \\ 0 & e^{-inkd} \end{pmatrix}$	$\frac{1}{n_1+n_2}\begin{pmatrix} 2n_1 & n_2-n_1 \\ n_1-n_2 & 2n_2 \end{pmatrix}$
M	$\begin{pmatrix} e^{-inkd} & 0 \\ 0 & e^{inkd} \end{pmatrix}$	$\frac{1}{2n_2}\begin{pmatrix} n_1+n_2 & n_2-n_1 \\ n_2-n_1 & n_1+n_2 \end{pmatrix}$

Table A.1: Multilayer matrices. The second column displays the propagation matrices through a niform dielectric of index n and thickness d. The third column show matrices for an interface between two media of index n_1 and n_2.

It can be verified that energy conservation and the phase relations derived in this appendix are automatically satisfied by applying the matrix procedure. The matrix calculation can also be applied to the Fabry-Perot to derive Eqs. (2.34) and (2.35).

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