Laser Physics 464 — Homework 2

Due Monday, September 25, 2023

Fourier Transform review

Each question 5 pts except 10 pts for question 2

1. Fluorescence decay

An atomic system has been prepared in the upper state, and starts fluorescing at $t = 0$. We expect an exponential decay. The electric signal amplitude is $\mathcal{E}(t) = e^{-at}u(t)$ where "a" is positive real number and

$$\begin{aligned} u(t) &= 0 \qquad\qquad t < 0 \\ &= 1 \qquad\qquad t \geq 0. \end{aligned} \tag{1}$$

a) Find the Fourier transform of the signal, in the frequency domain. Do you expect it to be real?

b) Sketch the amplitude and phase of $E(\Omega)$

Shape: real part is a Lorentzian, Imaginary part the corresponding dispersive curve.

$$E(\Omega) = \int_0^\infty e^{-(a+i\Omega)t} dt = \frac{1}{a + i\Omega} \tag{2}$$

2. Applying properties of linearity, shift to calculate Fourier transforms

Using the linearity property of the Fourier transform and the fourier transform of the square function, find the Fourier transform of the function defined by:

$$\begin{aligned} E(t) &= 0 \qquad\qquad t < 1 \\ &= 1 \qquad\qquad 1 < t < 2 \end{aligned}$$

$$
\begin{aligned}
&= 2 \qquad && 2 < t < 3 \\
&= 1 \qquad && 3 < t < 4 \\
&= 0 \qquad && t > 4
\end{aligned} \tag{3}
$$

We construct the Fourier transform starting from a square pulse of amplitude A, centered at zero, width 1, which is the sinc:

$$E_a(\Omega) = a\frac{\sin\frac{\Omega}{2}}{\frac{\Omega}{2}} \tag{4}$$

The three square pulses, if centered at zero, have as Fourier transform:

$$E_2\Omega + E_1(\Omega)[2\cos\Omega] \tag{5}$$

The ensemble being shifted by 2.5 time units:

$$E(\Omega) = 2\left\{(1+\cos\Omega)\frac{\sin\frac{\Omega}{2}}{\frac{\Omega}{2}}\right\}e^{2.5i\Omega}. \tag{6}$$

3. Correlation and Convolution

Find the autocorrelation and autoconvolution of the function defined by:

$$
\begin{aligned}
E(t) &= 0 \qquad && t < 0 \\
&= 1 \qquad && 0 < t < 1 \\
&= 0.5 \qquad && 1 < t < 2 \\
&= 0 \qquad && t > 2
\end{aligned} \tag{7}
$$

The time signal is a square pulse followed by a rectangle of height 1/2.

The autocorrelation is symmetric and centered at zero delay. The autocorrelation of the first square is a triangle of height 1, between -1 and +1. The autocorrelation of the second rectangle is a triangle of the same width but height 1/2. The sum is a triangle of height 1.5 between the same limits. To this we have to add the cross correlation of the large and small rectangles, which is composed of two triangles also of base 2, height 1/2, centered at +1 and -1.

4. Fourier transform of a triangle

Find the Fourier transform of an isosceles triangle. (Hint: think autocorrelation) sinc^2 since it is the autocorrelation of a rectangle.

5. Convolution

Polarization is not always an instantaneous response like assumed in $P(t) = \epsilon_0 \chi(t)\mathcal{E}(t)$ and the correct way to find the polarization is

$$P(t) = \epsilon_0 \int \chi(\tau)\mathcal{E}(t-\tau)d\tau \tag{8}$$

Where the electric field is a single pulse given as

$$\mathcal{E}(t) = e^{\frac{-t^2}{\tau_G^2}} \tag{9}$$

and the dielectric susceptibility is:

$$\begin{aligned} \chi &= 0 \qquad\qquad t < 0 \\ &= e^{\frac{-t}{\tau}} \end{aligned} \tag{10}$$

What is the polarization as a function of frequency? Assuming the polarization corresponds to an absorption line, how does the real part of the polarization affect the transmitted pulse? Shape: real part is a Lorentzian, Imaginary part the corresponding dispersive curve.

$$E(\Omega) = \int_0^\infty e^{-(a+i\Omega)t}dt = \frac{1}{a+i\Omega} \tag{11}$$

6. Time-Bandwidth Product

A Gaussian (chirped) pulse has the field amplitude:

$$\tilde{\mathcal{E}} = \mathcal{E}_0 e^{-(1+ia)(\frac{t}{\tau_G})^2}. \tag{12}$$

Disregarding the amplitude factors, show that the Fourier transform of the chirped Gaussian is:

$$\tilde{\mathcal{E}}(\Omega) = e^{-\Omega^2\tau_G^2/4(1+a^2)}. \tag{13}$$

Find the time bandwidth product for the Gaussian pulse if

(a) The width is defined by the FWHM of the intensity — find $\tau_{FWHM} \times \Omega_{FWHM}$.

(b) The width is defined by the mean square deviation (MSQ). Find $\langle \tau_{MSQ} \rangle \times \langle \Omega_{MSQ} \rangle$.

Derive the FWHM of the pulse intensity.

$$\tau_{FWHM} = \tau_G \sqrt{2 \ln 2}. \tag{14}$$

The time for which half of the peak intensity is reached is given by:

$$-\ln 2 = -\frac{2t_{HW}^2}{\tau_G^2} \tag{15}$$

The FWHM is twice t_{HW}, or:

$$\tau_{FWHM} = \tau_G \sqrt{2 \ln 2}. \tag{16}$$

If we re-write the Gaussian of Eq. (13) as:

$$\tilde{\mathcal{E}}(\Omega) = e^{-\Omega^2/W^2}, \tag{17}$$

we know from the previous derivation that:

$$\Omega_{FWHM} = W\sqrt{2 \ln 2} = 2\frac{\sqrt{1+a^2}}{\tau_G}\sqrt{2 \ln 2} \tag{18}$$

The time-bandwidth product in terms of the FWHM is thus:

$$4 \ln 2 \sqrt{1+a^2} \tag{19}$$

To define the width by MSQ, we note that $\langle t \rangle = \langle \Omega \rangle = 0$ because the pulses are symmetric and centered on the origin. We have thus only to calculate:

$$\begin{aligned} \langle t^2 \rangle &= \frac{\int_{-\infty}^{\infty} t\tilde{\mathcal{E}}(t) t\tilde{\mathcal{E}}(t)^* dt}{\int_{-\infty}^{\infty} |\tilde{\mathcal{E}}(t)|^2 dt} = \frac{\left(\frac{\tau_G}{\sqrt{2}}\right)^3 \int_{-\infty}^{\infty} x^2 e^{-x^2} dx}{\left(\frac{\tau_G}{\sqrt{2}}\right) \int_{-\infty}^{\infty} e^{-x^2} dx} \\ &= \frac{\tau_G^2}{2} \frac{\sqrt{\pi}/2}{\sqrt{\pi}}, \end{aligned} \tag{20}$$

and a similar expression in the frequency domain, except that τ_G is replaced by W given above. This leads to:

$$\langle t^2 \rangle \langle \Omega^2 \rangle = \frac{1}{4} \tag{21}$$

7. The poor man's attosecond source

A smart researcher created a train of ultrashort pulses by adding the to radiation of a Nd:YAG laser its second harmonic, third harmonic, fourth harmonic and fifth harmonic in phase. Assume all these harmonics have been reduced to the same amplitude. The Nd:YAG laser has a wavelength of 1.06 μ m. The nth harmonic of E has the field E^n.

1. What is the resulting field in time?
2. Assuming the 5 beams are made of pulses 2 ns wide, how does this affect the spectrum?

The total field is:

$$E_t = \sum_{j=1}^{j=5} E^j$$

.

$$(E-1)E_t = E^6 - E$$

The sum of this geometric series is thus:

$$E_t = \frac{E^{3.5}(E^{2.5} - E^{-2.5})}{E^{0.5}(E^{0.5} - E^{-0.5})}$$

We have assumed equal amplitude for all frequencies. Therefore, we are only concerned with the harmonic dependence.

$$E_t(t) \propto e^{3i\omega t} \left[\frac{\sin 2.5\omega t}{\sin 0.5\omega t}\right]$$

The central frequency is 3 ω. The first zero of the amplitude function is given by $2.5\omega\tau = \pi$, or $\tau = \pi/(2.5\omega) = \pi\lambda/(5\pi c) \approx 0.5$ fs.
The signal consists in a train of 0.5 fs pulses separated by $2\pi/\omega$.

In the Fourier domain, the spectrum is restricted to the region corresponding to the Fourier Transform of a 2 ns pulse.